ECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 63 NUMBER 3

PUBLIC LIBRARY
MAR 20 1956
PLYMOUTH

CONTENTS

Statistical Decisions . . . . . E. S. KEEPING 147
On a Class of Determinants Associated with a Matrix . . . . . M. P. EPSTEIN 160
Diophantus of Alexandria . . . . . J. D. SWIFT 163
On the Zeros of a Cubic Recurrence . . . . . M. F. SMILEY 171
Mathematical Notes . . . . . M. MARCUS, C. M. PETERSEN, E. A. NORDHAUS AND J. W. GADDUM, N. C. PERRY AND J. C. MORELOCK 173
Classroom Notes . . . . . C. A. HAYES, JR., A. W. WALKER, KULDIP SINGH, C. B. READ AND FERNA WRESTLER 180
Elementary Problems and Solutions 186
Advanced Problems and Solutions 190
Recent Publications 196
Obituary, William Deweese Cairns . . . . . W. B. CARVER 204
News and Notices 205
The Mathematical Association of America 210
The Thirty-ninth Annual Meeting of the Association 210
Itineraries of Visiting Lecturers 212
Officers and Committees as of January 1, 1956 214

MARCH 1956

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	E. P. STARKE
HOWARD EVES	L. M. KELLY	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	H. S. ZUCKERMAN
	GEORGE SPRINGER	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to R. W. MacDowell, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, W. L. Duren, Jr., University of Virginia
First Vice-President, R. V. Churchill, University of Michigan
Second Vice-President, G. B. Price, University of Kansas
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, C. B. Allendoerfer, University of Washington

Additional Members of the Board of Governors: M. A. Basoco, H. W. Brinkmann, H. H. Downing, R. M. Foster, Philip Franklin, E. R. Heineman, F. F. Helton, M. S. Hendrickson, A. S. Householder, R. C. Huffer, Ralph Hull, C. A. Hutchinson, S. B. Jackson, L. W. Johnson, P. S. Jones, F. W. Kokomoor, R. E. Langer, L. L. Lowenstein, Saunders MacLane, K. O. May, E. J. McShane, E. B. Miller, W. H. Myers, Ivan Niven, C. O. Oakley, E. N. Oberg, Morris Ostrofsky, J. F. Randolph, C. B. Read, F. A. Rickey, M. F. Smiley, C. E. Springer, G. B. Thomas, Jr., C. W. Trigg, M. A. Zorn

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

STATISTICAL DECISIONS*

E. S. KEEPING, University of Alberta

The name of Abraham Wald, who died in an airplane crash in 1950, will always be associated with statistical decision theory, which ranks as one of the three greatest advances in mathematical statistics within comparatively recent times. The first of these advances, started by W. S. Gosset (better known under the familiar pen-name of Student) but due above all to the grand old man among living statisticians, Sir Ronald Fisher, was the development of small sample theory. Exact distributions of the mean, the variance, the correlation coefficient and other statistics, at least for samples from certain types of populations, were obtained, tables were computed, and the experimental statistician was armed with a whole set of techniques, *t*-tests, *F*-tests, analysis of variance, *etc.*, which he has been happily applying ever since.

The second advance, associated particularly with J. Neyman and E. S. Pearson, was the theory of estimation and the testing of hypotheses. The concept of a confidence interval was developed. The statistician was regarded as interested primarily in testing a null hypothesis, such as the hypothesis that two observed samples come from normal populations with the same mean, against some reasonable alternative hypothesis. One of the most important ideas that grew up in this second period was that of the power of a test. A practical development of this theory was the compilation of tables for finding the sizes of samples necessary to discriminate in certain cases, at a given level of confidence, between one hypothesis and another.

Wald's theory of statistical decision functions [1] began a third era in statistical analysis and interpretation. Because the theory was initially developed with great mathematical rigor, so that most applied statisticians found the going pretty rough, the theory did not at first make much headway except in rather exclusive circles. Lately, however, several new books [2] and even two popularizations [3] have appeared, but it is still broadly true that the theory is hardly in a form usable by the practicing experimental statistician. The operations research analysts are, however, vigorously pursuing practical aspects of decision theory. It is with the use of decision theory in statistics that I am particularly concerned to-day, and I want to explain the underlying ideas as simply as I can.

Any statistician, and this includes any experimenter who makes measurements or observations on some quantity subject to random variation, is faced at times with the necessity of making a decision, even if the decision is only what value to adopt for the observed variable. In sequential sampling of a consignment of manufactured articles, the decision may be to recommend either the acceptance or rejection of the lot, or, as a third alternative, to do some more sampling. The practical man would therefore like a useful principle to guide his

* Presented to the Pacific Northwest Section of the Association on June 17, 1955.

choice of a decision, but the application of any such principle must depend largely on the circumstances of the particular problem and often will involve numerical constants which are known only very roughly.

Wald set up a principle which in most cases seems reasonable, even though in some special problems it recommends a course of action which common sense would scarcely approve. A wrong decision will as a rule entail a loss, and that rule which minimizes the expected loss could be regarded as the best. The risk of making a wrong decision can often be reduced to the vanishing point by taking enough observations, but experiments usually cost money, and the risk function which Wald introduced depends on both the cost of experimentation and the cost of making a wrong decision, estimates of these costs being supplied by the experimenter. In order to calculate the expected value of the total loss it is also necessary to know something about the *a priori* probabilities of the various possible states of Nature which might be connected with the different decisions, in particular the various statistical populations which the experimenter might be regarded as sampling, and this sort of information is hard to get.

If we could calculate the total risk under all decision rules, we would naturally choose the rule which minimizes this risk, and this would clearly be an optimum procedure. To avoid the difficulty of unknown *a priori* probabilities there are different methods open to us. One is the way adopted, though apparently with some misgivings, by Thomas Bayes, and later used by Laplace—to assume that, in the absence of any other information, the *a priori* probabilities are all equal. This is the so-called Principle of Insufficient Reason, which was vigorously attacked by Fisher. Wald's method was the *Minimax Principle*, the principle of choosing a rule which minimizes the maximum risk which could occur, whatever the unknown state of Nature might be. This amounts to the rather pessimistic attitude of expecting the worst and acting accordingly, a rule which often turns out to be wise. It is not surprising, however, that various modifications of the minimax principle seem more reasonable in particular circumstances. The general theory of decisions does not stand or fall by the minimax principle.

Statistical decision theory is closely tied up with the theory of games, another topic which has been actively pursued in recent years [4]. The theory of games has obvious applications in warfare and in certain economic situations, but it has lately become apparent that many statistical problems can be regarded as a sort of game, a two-person game which the statistician plays with Nature. His strategy consists in making certain decisions, before or after conducting a series of experiments or trials. Nature, of course, cannot be considered as a conscious opponent who can take advantage of mistakes made by the statistician, but she has a wealth of possibilities in any situation and the statistician wants to know which of these possibilities is the one that actually obtains. In a two-person game against an intelligent opponent the minimax rule is quite reasonable—

play as if your opponent could always be counted on to make the best possible moves in the circumstances, but in games against Nature it is often unduly pessimistic.

In many games the decisions regarding strategies are clear-cut; in such-and-such circumstances one will do so-and-so. These are *pure* decisions. In other games the choice between decisions may depend on a random process like dealing cards or rolling a die. This implies that there is, at a certain stage of the game, a probability distribution over the set of possible decisions, with probabilities governed by the randomizing device, and such decisions are called *mixed*. In the children's game of Snakes and Ladders all moves are mixed, the decision to move a certain number of places, from 1 to 6, depending entirely on the chance throw of a die. In the game of the statistician against Nature, a mixed strategy for Nature is an *a priori* probability distribution of the parameters in a law.

Although in practice two-person games consist of alternate moves by the players, yet in theory each player could formulate his complete strategy for the whole game ahead of time, stating what counter-move he would make to each possible move by his opponent. In chess, White has an initial choice of 20 legal opening moves and makes a decision. Black replies with one of 20 moves. White's second move may depend on which of the 20 Black chooses, and his decision could be specified for each case. It is clear, however, that any actual enumeration of strategies for a whole game is not feasible, except in the very simplest games such as ordinary tic-tac-toe. Yet it is very convenient to think of a game, in the general sense, as a product of two spaces, a space X of all possible strategies by player I, and a space Y of all possible strategies by player II, together with a function $M(x, y)$ which specifies for each x belonging to X and each y belonging to Y what the pay-off will be to player II. Of course there is also a pay-off to player I, but in zero-sum games, with which we are concerned, the total pay-off is zero, so that player I's pay-off is $-M(x, y)$.

If player I uses a mixed strategy, choosing x_i with probability λ_i, then $x = \sum_i \lambda_i x_i$ is an element of X, and similarly $y = \sum_j \mu_j y_j$ is an element of Y. For each pair of strategies x_i, y_j, there will be an outcome r, depending on the rules of the game, or, if there are chance moves in the game, there will be a probability $p_{ij}(r)$ for the outcome r. Associated with each r is also a bounded numerical function $u(r)$ which may be regarded as the monetary gain to player II (or some monotone increasing function of this gain), supposing that the outcome r occurs.

The pay-off to player II is the expected value of $u(r)$, namely

$$M_{ij} = M(x_i, y_j) = \sum_{r \in R} p_{ij}(r) u(r), \tag{1}$$

where R is the space of all possible outcomes.

The game may thus be considered as a matrix (M_{ij}) in which the columns represent strategies of player I, the rows strategies of player II, and the (i, j)th

element the pay-off $M(x_i, y_j)$. Player I chooses a column

		I			
		x_1	x_2	$\cdots$	x_m
II	y_1	M_{11}	M_{21}	$\cdots$	M_{m1}
	y_2	M_{12}	M_{22}	$\cdots$	M_{m2}
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	y_n	M_{1n}	M_{2n}	$\cdots$	M_{mn}

and player II simultaneously chooses a row, neither knowing the other's choice. The pay-off to II is the corresponding entry in the matrix. If either player uses a mixed strategy, he chooses two or more pure strategies with probabilities λ_i or μ_j.

It is not difficult to devise particular pay-off schemes in which different criteria for the best strategy lead to different decisions [5]. An example is the following:

	x_1	x_2	x_3	x_4
y_1	2	2	0	1
y_2	1	1	1	1
y_3	0	4	0	0
y_4	1	3	0	0

The Laplace method assumes that Player I is equally likely to employ any one of his four strategies (in the game of the statistician against Nature, the *a priori* probabilities for all values of an unknown parameter are assumed equal). The expectation of II for strategy y_j is the average gain $(\sum_i M_{ij})/m$, and he should choose the row which maximizes this. In the example, these values are 5/4, 1, 1, 1, so that he should choose y_1.

The minimax principle of Wald states that the player should choose that row for which his smallest possible gain is a maximum. In the four rows of the example the minimum gains are 0, 1, 0, 0, so that player II should prefer strategy y_2. Whatever happens he cannot then gain less than one unit.

Hurwicz [7] has modified Wald's criterion so as to remove some of its pessimistic character. He suggests selecting a number $\alpha(0 \leqq \alpha \leqq 1)$ which may be regarded as measuring player II's optimism. For each row (or probability mixture of rows when mixed strategies are allowed), let m_i be the least value of M_{ij} and M_i the greatest. Player II should choose that row for which $\alpha M_i + (1-\alpha)m_i$ is a maximum. When $\alpha = 0$ this clearly reduces to Wald's criterion. In the example above, the four rows give 2α, $\alpha+(1-\alpha)$, 4α, 3α respectively, so that, as long as $\alpha > \frac{1}{4}$, strategy y_3 is to be preferred to the others.

It was pointed out by Savage [6] that it is often reasonable to minimize what

he called the *maximum regret*. Player II's regret is the difference between the pay-off for his actual strategy y_j and the pay-off he might have got if he had known beforehand what strategy x_i player I would adopt, in which case he would naturally have selected that strategy for which M_{ij} (given i) is a maximum. The regret may be defined as

$$r_{ij} = \max_j M_{ij} - M_{ij}. \tag{2}$$

Player II then chooses the y_j which minimizes the maximum over all i of r_{ij}.

In the example above, when $i=1$, max $M_{ij}=2$, so that $r_{11}=0$, $r_{12}=1$, *etc.* The regret matrix is

	x_1	x_2	x_3	x_4
y_1	0	2	1	0
y_2	1	3	0	0
y_3	2	0	1	1
y_4	1	1	1	1

The maxima of r_{ij} for the different rows are 2, 3, 2, 1, so that on Savage's criterion player II would choose strategy y_4. He is sure then of not missing the boat by more than 1 unit.

Four different criteria lead therefore in this example to four different decisions regarding the choice of strategy. No strategy here dominates all the others in the sense of being at least as good whatever player I does, and definitely better for at least one possibility.

It has been objected to the Savage criterion that if a new strategy is added to the old set (an extra row in the matrix) the principle of minimax regret may dictate a different strategy from the one previously preferred, even though both of these were in the original set. That is, the relative order of preference among the y_j may be altered by an additional y. For example, if to the set

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

we add a third row $(-7\ 4)$, the regret matrix changes from

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \text{ to } \begin{pmatrix} 0 & 3 \\ 2 & 2 \\ 9 & 0 \end{pmatrix}$$

and the maximum regret per row changes from

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ to } \begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}.$$

The principle of minimizing this maximum regret would therefore suggest using strategy 1 *before* the addition of strategy 3, but strategy 2 *afterwards*. However, all the practicable criteria so far devised are subject to somewhat similar criticisms [8].

The above discussion assumes that in the kind of games played, or the kind of statistical decisions made, there is in fact a numerically measurable pay-off, if not in actual dollars and cents, at least in what the economists call "utility." A dollar does not mean the same thing to a rich man and to a poor man, or in other words the utility of a unit of money appears to diminish as one's wealth increases, being perhaps inversely proportional to this wealth (although this is not true of inveterate gamblers who will often gamble more wildly, the less money they have). There has been much argument regarding the validity of the theory of utility, and the major criticisms are dealt with by Savage [2], and by Arrow [11]. In the statistical field, numerical values can very seldom be assigned to the consequences of decisions, except in the sense that a "true" decision (one that is confirmed by subsequent research) is certainly preferable to a false one, so that one could reasonably associate a certain unit loss with a wrong decision and a zero loss with a correct decision. The statistician protects himself to some extent by making assertions which are, as far as he can tell, unlikely to be in error more than a known fraction (say 5%) of the time.

In the game of the statistician against Nature, there will usually be one or more parameters ω, belonging to a parameter space Ω, which the statistician desires to estimate. The possible values of ω are the pure strategies for player I (Nature), and an *a priori* probability distribution for ω is a mixed strategy. In game theory the spaces X and Y are finite, but in statistics the space Ω is usually infinite.

The statistician (player II) performs experiments to spy on Nature. The outcome of an experiment will be a value of z belonging to the sample space Z. An event is a set S of values of z and will have a probability, depending on ω, given by

$$P_\omega(S) = \sum_{z \in S} p_\omega(z). \tag{3}$$

As a result of the experiments the statistician will take an action a belonging to a set A of possible actions. The function $d(z) = a$ which maps the space Z onto A is a *statistical decision function*. The space Z may be supposed divided into disjoint sets S_a, each set corresponding to one action a, so that if z falls in S_a, the action a is taken. A familiar example is the ordinary one-tailed t-test, in which the space is the real axis, divided into two regions, $t > t_a$ and $t < t_a$. If the observed t falls in the former region the null hypothesis is rejected, and if in the latter region it is accepted. The class D of all possible decisions is the class of pure strategies for the statistician. If he flips a coin to decide which of two decisions to adopt, that is a mixed strategy. The class D is often infinite.

The statistician does not usually stand to win anything tangible in his game

with Nature, but he may lose. Experiments cost money or time or both, and a decision based on a false estimate of the parameter ω may be expensive. Although in practice it is often difficult or impossible to estimate with any accuracy the loss which may be caused by wrong decisions, we can suppose for the sake of theory that there is a numerical loss, the magnitude of which, $L(d, \omega)$, depends on the true value ω and the decision d. The pay-off function for the statistician's game is the expected value of this loss, namely

$$M(d, \omega) = \sum_z L(d, \omega) p_\omega(z), \tag{4}$$

in which $d = d(z)$. He should try to act so as to minimize the expected loss.

Occasionally the minimax principle will suggest a different estimate from the customary one. Suppose we wish to estimate the parameter θ (the probability of success), for a binomial distribution, on the basis of our observation that the event has occurred x times in n trials. The usual estimate is, of course, x/n. This estimate is unbiased, since the expectation of x/n, given θ, is precisely θ. It is also the maximum likelihood estimate, since the probability of x/n, given θ, is a maximum when $\theta = x/n$. But it is not the minimax estimate.

If our estimate of θ is $f_n(x)$, our loss may be regarded as proportional to the square of the difference between $f_n(x)$ and the true value. This seems reasonable since, if there were a linear term, the loss would change sign with $f_n(x) - \theta$, so that an error in one direction would represent a gain. The expectation of loss is, for a given θ,

$$kE\{f_n(x) - \theta\}^2 = k \sum_{x=0}^{n} \{f_n(x) - \theta\}^2 p_n(x), \tag{5}$$

where

$$p_n(x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}. \tag{6}$$

If $\lambda(\theta) d\theta$ is the probability, before any trials are made, that θ lies in the interval θ to $\theta + d\theta$, the expectation of loss is

$$L = \int_0^1 kE\{f_n(x) - \theta\}^2 \lambda(\theta) d\theta. \tag{7}$$

An estimate which minimizes this is called a *Bayes estimate*. It may be shown that this estimate is equal to the expectation of θ, given x, which can be calculated by Bayes' theorem if $\lambda(\theta)$ is known. If for example $\lambda(\theta)$ is constant, $f_n(x)$ turns out to be $(x+1)/(n+2)$, an estimate which would minimize our expected loss if we were totally ignorant beforehand of the value of θ, but knew that all possible values were equally likely.

A Bayes estimate is *minimax* if the expectation of loss for a given θ is the same whatever the value of θ. This is so for the estimate $f_n(x) = (x + \frac{1}{2}\sqrt{n})/(n + \sqrt{n})$, which is also a Bayes estimate when $\lambda(\theta) = c\theta^a(1-\theta)^a$, where $a = \frac{1}{2}\sqrt{n} + 1$. The

minimax estimate of θ is therefore $(x+\frac{1}{2}\sqrt{n})/(n+\sqrt{n})$. If for instance we made four trials and observed one success, the ordinary estimate of θ would be $\frac{1}{4}$. The minimax estimate would be $\frac{1}{2}$. The loss for the ordinary estimate is $k\theta(1-\theta)/4$, and that for the minimax estimate is $k/36$. For any value of θ between 0.13 and 0.87 the minimax estimate is the better.

In the field of quality control, there is a practical need for decisions. A factory, for example, buys large numbers of small parts which come in lots and may contain a certain proportion θ of defectives. Complete inspection of the lots is costly, and no inspection at all may permit very poor lots to get by, with resultant trouble and expense to the factory.

Suppose that the parts come in lots of N, that the cost of inspecting an item is c_1 and that of scrapping a defective item is c_2, which is presumably much greater than c_1. There are two possible decisions, d_1 (to accept the lot) and d_2 (to inspect it 100%), the costs of these actions being $N\theta c_2$ and Nc_1 respectively, since with complete inspection no defectives will be purchased. If $\lambda(\theta)$ is the *a priori* probability density for θ, and if the probabilities for d_1 and d_2 are μ and $1-\mu$ respectively, the expected loss is

$$L(\lambda, \mu) = \int_0^1 \lambda(\theta)\{\mu N\theta c_2 + (1-\mu)Nc_1\}d\theta \tag{8}$$

$$= Nc_1 + N\mu\int_0^1 \lambda(\theta)(\theta c_2 - c_1)d\theta.$$

This obviously depends upon λ as well as μ. In practice there will usually be some previous experience to draw upon in estimating λ. If there is no such information the minimax principle suggests the unreasonable strategy $\mu=0$ (that is, never accept a lot without complete inspection), since there is a possibility that $\lambda(\theta)$ might be 0 for $\theta<c_1/c_2$, and $L(\lambda, \mu)$ is then minimized for $\mu=0$. The alternative assumption that $\lambda(\theta)$ is constant gives $L(\lambda, \mu)=Nc_1+\frac{1}{2}N\mu(c_2-2c_1)$, so that the minimum loss is given for $\mu=0$ if $c_2>2c_1$ but for $\mu=1$ if $c_2<2c_1$.

One solution of the difficulty is to inspect n items out of a lot and decide to accept the lot if the number of defectives is less than s but to resort to complete inspection if $x \geqq s$. This amounts to putting

$$\mu = \sum_{x=0}^{s-1} p_n(x) = 1 - P(s, n, \theta), \tag{9}$$

where $P(s, n, \theta)$ is the cumulative binomial probability function, now extensively tabulated [9].

The cost of decision d_1 is $nc_1+(N-n)\theta c_2$, so that the expected loss is

$$L(\lambda, s) = Nc_1 + (N-n)\int_0^1 \lambda(\theta)(c_2\theta - c_1)[1 - P(s, n, \theta)]d\theta. \tag{10}$$

The minimax principle still advises 100% inspection whatever the result of

the sampling. The Laplace principle, $\lambda(\theta) = \text{constant}$, advises a choice of s as near as possible to the value of $(c_1(n+2)/c_2) - \frac{1}{2}$. For $c_2/c_1 = 25$, $N = 1000$, $n = 100$, this gives $s = 4$. The advice is to accept a lot if in a sample of 100 there are not more than three defectives.

Any solution which minimizes $L(\lambda, s)$ for a given $\lambda(\theta)$ is a Bayes solution, and in this problem any Bayes solution boils down to a choice of the integer s. If for example $\lambda(\theta) = 25$ for $0.01 < \theta < 0.05$ and $\lambda(\theta) = 0$ for all other values of θ (that is, θ has a rectangular distribution over an interval of length 0.04), the loss is given by

$$L(s) = Nc_1 + \frac{25(N-n)}{n+1}\left\{\sum_{x=0}^{s-1}\frac{c_2(x+1)}{n+2}\left[P(x+2, n+2, 0.05) - P(x+2, n+2, 0.01)\right] - c_1\sum_{x=0}^{s-1}\left[P(x+1, n+1, 0.05) - P(x+1, n+1, 0.01)\right]\right\}. \tag{11}$$

For the data mentioned above, the minimum loss is given by $s = 7$. With the fuller information now available on θ it is economical to accept a lot if there are not more than six defectives in a sample of 100.

This problem can also be treated from the point of view of minimax regret. If we suppose that there is in fact a true, but unknown, probability θ for the occurrence of defectives in the lots supplied by the manufacturer, the loss suffered by the purchaser is, per lot,

$$L(s, \theta) = Nc_1 + (N-n)(c_2\theta - c_1)\left[1 - P(s, n, \theta)\right] \tag{12}$$

for $s \geqq 1$.

The regret function is the difference between the actual loss and the minimum loss that would be incurred even if the true value of θ were known. Symbolically,

$$r(s, \theta) = L(s, \theta) - \inf_{s} L(s, \theta). \tag{13}$$

For each possible s, one can calculate what value of θ would make $r(s, \theta)$ a maximum, and then decide on the particular s which would minimize this maximum regret. For the data given above the calculations suggest that the best strategy in Savage's sense is to take $s = 4$, that is, to accept without inspection a lot with not more than three defectives in a random sample of 100. This example illustrates how the wise decision in such a case may depend not only on the extent of the information available on the value of the unknown parameter but also on the criterion adopted for judging the best decision.

Wald's most well-known contribution to statistical theory is undoubtedly the sampling procedure called *sequential analysis*. The idea was that observations should not all be taken at once, before being analyzed, but that they should be accumulated in successive stages. At each stage the statistician would have to

make a decision, either to take some definite action or to continue with the observations, this decision depending on all the observations available up to that stage. Wald set up a decision rule, the *sequential probability ratio test*, which on certain reasonable assumptions turned out to be remarkably efficient in economizing observations, and which has been quite widely adopted in practice.

We will consider a simplified problem to which sequential analysis is applicable. Suppose we have to decide between two alternative hypotheses H_1 and H_2 of which the *a priori* probabilities are g and $1-g$ respectively. The loss incurred in accepting H_2 if H_1 is true is w_{12}, and the loss in accepting H_1 if H_2 is true is w_{21}. (It is assumed that there is no loss in accepting a *true* hypothesis.) The cost of a single observation is a constant c, which we may take as unity, and all the observations are supposed independent of each other and subject to a common distribution law. The hypotheses concern the value of a parameter p in this law, H_1 being that $p=p_1$ and H_2 that $p=p_2$.

If g is sufficiently small it will be wise to accept H_2 without more ado, and run the risk $R_1=w_{12}g$. Similarly, if g is sufficiently large, it will be wise to accept H_1 with risk $R_2=w_{21}(1-g)$. The whole interval $0 \leqq g \leqq 1$ can be divided into three parts, $I_1(0 \leqq g \leqq \underline{g})$, $I_2(\underline{g} < g < \bar{g})$, and $I_3(\bar{g} \leqq g \leqq 1)$, such that if g belongs to I_1 we accept H_2, if to I_3 we accept H_1, but if to I_2 we take one or more observations. In this case there will be a sequential test with an average risk less than either R_1 or R_2.

If we take n observations, the *a posteriori* probability of H_1 will be given by Bayes' rule:

$$g_n = gP_{1n}/[gP_{1n} + (1-g)P_{2n}]. \tag{14}$$

where P_{1n} is the probability of the observed set of results on H_1, and P_{2n} the probability of the same set on H_2. The experimentation is stopped as soon as g_n falls either in I_1 or in I_3, the decision being to accept H_2 or H_1 in the respective cases. As long as g_n falls in I_2 the sampling continues.

Writing $W_n=P_{2n}/P_{1n}$, we can easily verify that this procedure is exactly the same as Wald's probability-ratio test, the ratio W_n being computed at each stage, and sampling continued as long as $B<W_n<A$, where A and B are two numbers, depending on g, $\underline{g}$ and $\bar{g}$,

$$A = \frac{1-\underline{g}}{\underline{g}} \cdot \frac{g}{1-g}, \qquad B = \frac{1-\bar{g}}{\bar{g}} \cdot \frac{g}{1-g}. \tag{15}$$

For convenience we can suppose that hypotheses H_1 and H_2 correspond to symmetric binomial dichotomies, p_1 being the proportion of "successes" under H_1, and p_2 the proportion under H_2, with $p_2=1-p_1=q_1$. We can take $p_1<p_2$. If s_n is the number of successes in n trials, and therefore $n-s_n$ the number of failures,

$$\frac{P_{2n}}{P_{1n}} = \frac{p_2^{s_n} q_2^{n-s_n}}{p_1^{s_n} q_1^{n-s_n}} = \left(\frac{q_1}{p_1}\right)^{2s_n-n}. \tag{16}$$

Let x be an observed variable, which takes only the values 1 for a success and -1 for a failure. Then, after n observations,

$$\sum_{i=1}^{n} x_i = s_n - (n - s_n) = 2s_n - n, \tag{17}$$

so that, from (16) and (17),

$$W_n = P_{2n}/P_{1n} = u^z,$$

where $u = q_1/p_1$, and $z = \sum x_i$. The probability ratio test is therefore equivalent to noting z after each observation and continuing as long as $-b < z < a$, where $-b = \log B/\log u$ and $a = \log A/\log u$. When $z \geqq a$ we accept H_2, and when $z \leqq -b$ we accept H_1. Since z will always be an integer, the test will end when $z = a'$ or $z = -b'$, where a' stands for the smallest positive integer $\geqq a$.

When $g = \underline{g}$, $A = 1$ and therefore $a = 0$. The risk of accepting H_2 without observations is $w_{12}\,\underline{g}$. This risk ought to be equal to that of the sequential test with $a' = 1$, $b' = b^*$ where b^* is the smallest positive integer greater than or equal to

$$-\log\left(\frac{1-\bar{g}}{\bar{g}} \cdot \frac{\underline{g}}{1-\underline{g}}\right) \Big/ \log u.$$

Similarly, if $g = \bar{g}$, we have $b = 0$, $(b' = 1)$ and $a' = b^*$, and the risk for this sequential test should be equal to $w_{21}(1-\bar{g})$.

If L_1 and L_2 are the probabilities of accepting H_1 as a result of a sequential test with limits a' and $-b'$, given that H_1 is true and that H_2 is true respectively, it can readily be shown that

$$\begin{aligned} L_1 &= (u^{a'+b'} - u^{b'})/(u^{a'+b'} - 1), \\ L_2 &= (u^{a'} - 1)/(u^{a'+b'} - 1). \end{aligned} \tag{18}$$

Also, if the expected numbers of observations required to lead to a decision in these two cases are E_1 and E_2 respectively,

$$\begin{aligned} E_1 &= [(a' + b')L_1 - a']/(q_1 - p_1), \\ E_2 &= [a' - (a' + b')L_2]/(q_1 - p_1). \end{aligned} \tag{19}$$

The risk involved in the sequential test is

$$R = g[E_1 + w_{12}(1 - L_1)] + (1 - g)[E_2 + w_{21}L_2]. \tag{20}$$

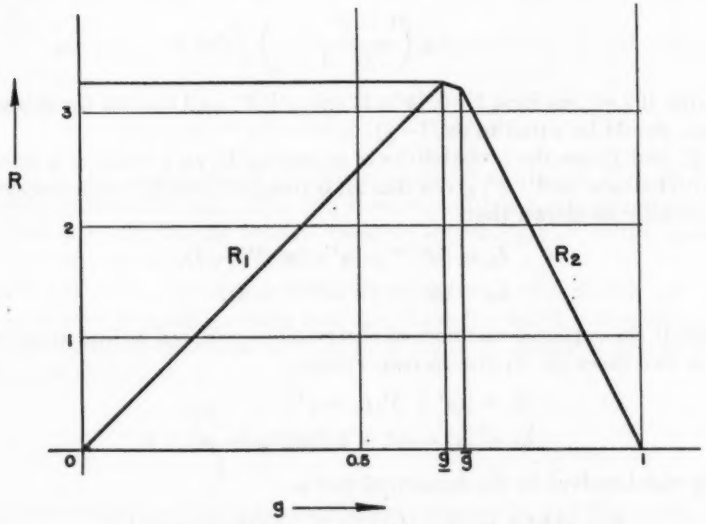
Since a' and b' are integers, the coefficients of g and $1-g$ will change abruptly at discrete values of g, so that the curve of R will consist of straight line segments. The values of $\underline{g}$ and $\bar{g}$ are given by

$$w_{12}\underline{g} = \underline{g}\left[\frac{(1+b^*)L_1 - 1}{q_1 - p_1} + w_{12}(1 - L_1)\right] + (1 - \underline{g})\left[\frac{1 - (1+b^*)L_2}{q_1 - p_1} + w_{21}L_2\right] \tag{21}$$

and

$$w_{21}(1 - \bar{g}) = \bar{g}\left[\frac{(b^*+1)L_1 - b^*}{q_1 - p_1} + w_{12}(1 - L_1)\right] + (1 - \bar{g})\left[\frac{b^* - (b^*+1)L_2}{q_1 - p_1} + w_{21}L_2\right], \tag{22}$$

where in (21) L_1 and L_2 are found by putting $a'=1$, $b'=b^*$ in (18) and in (22) by putting $a'=b^*$, $b'=1$. The method of solution is to guess a value of b^*, find $\underline{g}$ and $\bar{g}$ and see whether b^* then satisfies its definition. If we take, as a numerical example, $p_1=\frac{1}{3}$, $p_2=\frac{2}{3}$, $w_{12}=5$, $w_{21}=10$, we find the equations are satisfied by $b^*=1$, $\underline{g}=13/20$, $\bar{g}=17/25$. The curve (20) here consists of a single segment. Since x must be either 1 or -1, the sequential test will terminate after one observation.



The maximum risk is $R=13/4$, corresponding to $g=\underline{g}=13/20$. This is the best strategy for Nature.

The statistician employing the minimax principle can either accept H_2 with risk $5g$ or use the sequential test with limits 1 and -1 with risk given by (20).

These risks agree at $g=\underline{g}$ but not for other values. However, he can employ a mixed strategy, accepting H_2 with probability μ and doing the test with probability $1-\mu$, μ being so chosen that the total risk, $\mu(5g)+(1-\mu)R$, is independent of g. For the data used in the example above it turns out that $\mu=\frac{1}{4}$. The minimax strategy consists, then, in using some randomizing device, such as a table of random numbers, to decide either (with probability $\frac{1}{4}$) on accepting H_2 without taking any observations or (with probability $\frac{3}{4}$) on doing a sequential test with limits 1 and -1.

Further examples of dichotomies like the above, and also of trichotomies, in which the various *a priori* distributions are represented by regions in a triangle, are given by Arrow, Blackwell and Girshick [10].

It may be noted that the sequential process is analogous to a game between two players, A and B. At each trial either A pays B one dollar (probability p_1) or B pays A one dollar (probability p_2). A starts with capital a' dollars and B with capital b' dollars. The game ends when either A or B is ruined. The quantity z is the net amount paid by A to B. The probability that B will eventually be ruined is L_1, as given by (18).

References

1. A. Wald, Statistical Decision Functions, Wiley, 1950.

2. *e.g.* L. J. Savage, The Foundations of Statistics, Wiley, 1954, D. Blackwell and M. Girshick, Theory of Games and Statistical Decisions, Wiley, 1954, R. M. Thrall *et al.*, Decision Processes, Wiley, 1954.

3. I. D. J. Bross, Design for Decision, Macmillan, 1953, J. D. Williams, The Compleat Strategyst, McGraw-Hill, 1954.

4. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behaviour, Princeton, 2nd ed., 1947. J. C. C. McKinsey, Introduction to the Theory of Games, McGraw-Hill, 1952.

5. J. Milnor, Games against Nature, in the book Decision Processes (ref. 2).

6. L. J. Savage, The Theory of Statistical Decision, Journ. Amer. Stat. Assoc., vol. 48, 1947, pp. 238–248.

7. L. Hurwicz, A Class of Criteria for Decision Making under Ignorance, Cowles Commission Discussion Paper (mimeographed).

8. See ref. 5 and also R. Radner and J. Marschak, Note on some proposed Decision Criteria, in the same book.

9. Tables of Cumulative Binomial Probabilities, Ordnance Corps, Washington, D. C., 1952.

10. K. J. Arrow, D. Blackwell and M. A. Girschick, Bayes and minimax solutions of sequential decision problems, Econometrica, vol. 17, 1949, pp. 213–244.

11. K. J. Arrow, Alternative Approaches to the Theory of Choice in Risk-taking Situations, Econometrica, vol. 19, 1951.

ON A CLASS OF DETERMINANTS ASSOCIATED WITH A MATRIX

M. P. EPSTEIN, University of California, Berkeley

Let A be an $n \times n$ matrix with elements in a field F and let $g(x)$, the minimal polynomial of A, be of degree s. We shall say that A is diagonalizable if there is a non-singular $n \times n$ matrix P with elements in some extension field of F such that PAP^{-1} is in diagonal form.

Let Tr denote the trace function, and for each positive integer m define

$$\Delta_m = \det [\mathrm{Tr}\,(A^{i+j})] \qquad 0 \leqq i, j \leqq m-1.$$

There are several interesting relationships between determinants of this type and certain properties of the matrix A. For example, in a recent paper by H. Flanders and the author [1] the following theorem was proved:

THEOREM 0. *If F is of characteristic zero then A is diagonalizable if and only if $\Delta_s \neq 0$.*

Actually this theorem is a consequence of the following more general property of these determinants:

THEOREM 1. *Suppose A has exactly r different characteristic roots. Then:*

1. *If F is of characteristic zero then r is equal to the maximum value of m for which $\Delta_m \neq 0$;*
2. *If F is of characteristic p different from zero and if m is a positive integer such that $\Delta_m \neq 0$ then $m \leqq r$ (such values of m exist at least when n is not divisible by p). In particular, if $n < p$ then again r is equal to the maximum such value of m.*

Proof. Let $\lambda_1, \cdots, \lambda_r$ be the distinct characteristic roots of A of respective multiplicities $n_1, \cdots, n_r$ ($n_1 + \cdots + n_r = n$) and let V be the (non-vanishing) Vandermonde determinant of $\lambda_1, \cdots, \lambda_r$. Then, as in the proof of Theorem 0, we see that

$$\Delta_r = n_1 n_2 \cdots n_r V^2 \tag{1}$$

while for $m > r$ we have

$$\Delta_m = n_1 n_2 \cdots n_r \begin{vmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \lambda_1 & \cdots & \lambda_r & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{r-1} & \cdots & \lambda_r^{r-1} & 0 & \cdots & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_r & \cdots & \lambda_r^{r-1} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}$$

which vanishes since there are $m-r$ columns of zeros in the first determinant on the right and $m-r$ rows of zeros in the second.

The conclusions of the theorem follow from these equalities.

This theorem provides a computational method of evaluating the number r of distinct characteristic roots of a given matrix A. For, $r \leqq s \leqq n$, so that by utilizing the sequence $\{\Delta_1, \Delta_2, \cdots, \Delta_s\}$ if s has been previously computed, or the sequence $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ otherwise, one determines r as the maximum subscript for which a non-zero value is obtained.

There are certain inconveniences to this method however, in that there may exist integers m_1, m_2 with $m_1 < m_2$ such that $\Delta_{m_1} = 0$ but $\Delta_{m_2} \neq 0$. Therefore in using either sequence the computations cannot be ended as soon as a value of zero is obtained, but must be continued until the entire sequence of values has been evaluated.

However in particular cases this circumstance can be avoided. For example, suppose F is contained in the field of complex numbers. Let A^* be the transposed conjugate matrix of A and for each positive integer m define

$$\Delta_m^* = \det [\operatorname{Tr} (A^i A^{*j})] \qquad 0 \leqq i, j \leqq m - 1.$$

We then have:

THEOREM 2. *Suppose A is normal (i.e. $AA^* = A^*A$). If A has exactly r different characteristic roots then $\Delta_m^* \neq 0$ for $m \leqq r$ while $\Delta_m^* = 0$ for $m > r$.*

Proof. Let $\lambda_1, \cdots, \lambda_n$ be the characteristic roots of A. Then the characteristic roots of A^* are $\bar{\lambda}_1, \cdots, \bar{\lambda}_n$. Further we can consider A and A^* as defining linear transformations on an n-dimensional complex unitary vector space M. By the normality condition on A we see that M has an orthonormal base $x_1, \cdots, x_n$ such that $Ax_k = \lambda_k x_k$, $A^* x_k = \bar{\lambda}_k x_k$, $1 \leqq k \leqq n$. Thus for any non-negative integers, i, j, $A^i A^{*j} x_k = \lambda_k^i \bar{\lambda}_k^j x_k$ so that $\operatorname{Tr} (A^i A^{*j}) = \sum_{k=1}^n \lambda_k^i \bar{\lambda}_k^j$. Or, if $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A, with respective multiplicities $n_1, \cdots, n_r$ ($n_1 + \cdots + n_r = n$), we have $\operatorname{Tr} (A^i A^{*j}) = \sum_{k=1}^r n_k \lambda_k^i \bar{\lambda}_k^j$.

Now suppose $m \leqq r$, and define the $m \times r$ matrix

$$C_m = \begin{pmatrix} n_1^{1/2} & \cdots & n_r^{1/2} \\ n_1^{1/2}\lambda_1 & \cdots & n_r^{1/2}\lambda_r \\ \cdots & \cdots & \cdots \\ n_1^{1/2}\lambda_1^{m-1} & \cdots & n_r^{1/2}\lambda_r^{m-1} \end{pmatrix}.$$

We have then, as in the proof of Theorem 1

$$\Delta_m^* = \det (C_m C_m^*)$$

(noting that $C_m C_m^*$ is an $m \times m$ matrix since C_m is $m \times r$ and C_m^* is $r \times m$). But $C_m C_m^*$ is the Gramian matrix of the vectors $(n_1^{1/2}, n_1^{1/2}\lambda_1, \cdots, n_1^{1/2}\lambda_1^{m-1})$, $\cdots$, $(n_r^{1/2}, n_r^{1/2}\lambda_r, \cdots, n_r^{1/2}\lambda_r^{m-1})$ and therefore the rank of $C_m C_m^*$ is equal to the rank of C_m. Considering only the first m columns of C_m we obtain a matrix whose

determinant is equal to

$$(n_1 n_2 \cdots n_m)^{1/2} \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \cdots & \cdots & \cdots \\ \lambda_1^{m-1} & \cdots & \lambda_m^{m-1} \end{vmatrix},$$

which is different from zero since $\lambda_1, \cdots, \lambda_m$ are distinct. Thus rank $C_m C_m^*$ $=$rank $C_m = m$, so $\Delta_m^* = \det(C_m C_m^*) \neq 0$.

If $m > r$ an argument similar to that in the proof of Theorem 1 shows that $\Delta_m^* = 0$. This proves the theorem.

Returning now to the general situation where F is an arbitrary field, we have the following theorem (which includes Theorem 0) as a consequence of Theorem 1, and in particular, of equation (1).

THEOREM 3. *The determinants Δ_m have the following properties:*

1. *Δ_n is equal to the discriminant of the characteristic polynomial of A.*
2. *Suppose A has exactly r different characteristic roots $\lambda_1 \cdots, \lambda_r$ of respective multiplicities $n_1, \cdots, n_r (n_1 + \cdots + n_r = n)$ and let D be the discriminant of $g(x)$, a polynomial of degree s. Then $\Delta_s = n_1 n_2 \cdots n_r D$.*

Therefore:

3. *If F is of characteristic zero, a necessary and sufficient condition that A be diagonalizable is that $\Delta_s \neq 0$.*
4. *If F is of characteristic p different from zero, a sufficient condition that A be diagonalizable is that $\Delta_s \neq 0$ (and if $n < p$ this condition is again necessary).*
5. *For any characteristic of F, a necessary and sufficient condition that A have its characteristic roots all distinct is that $\Delta_n \neq 0$, and this condition is also sufficient for the diagonalizability of A.*

Proof. We shall prove assertion 2 first. Clearly $r \leqq s \leqq n$. If $r < s$ then $g(x)$ has a repeated root so $D = 0$. But then by Theorem 1 also $\Delta_s = 0$, so assertion 2 holds in this case. If $r = s$ then assertion 2 follows from equation (1) and the fact that the discriminant of a monic polynomial is equal to the square of the Vandermonde determinant of its roots. Again, if $r < n$ then both Δ_n and the discriminant of the characteristic polynomial of A are equal to zero. If $r = n$ then $n_1 = n_2$ $= \cdots n_r = 1$. But then we have also $s = n$, so assertion 1 follows from assertion 2.

To prove assertions 3 and 4 we recall that A is diagonalizable if and only if $g(x)$ has distinct roots, *i.e.*, if and only if $D \neq 0$. If $\Delta_s \neq 0$ this condition is satisfied. Conversely, suppose $D \neq 0$. If F is of characteristic zero then $\Delta_s \neq 0$. If F is of characteristic p different from zero then the condition $n < p$ implies that $n_i < p$ for $i = 1, \cdots, r$, so again $\Delta_s \neq 0$.

Assertion 5 is immediate.

Bibliography

1. M. Epstein and H. Flanders, On the reduction of a matrix to diagonal form, this MONTHLY, vol. 62, 1955, pp. 168–171.

DIOPHANTUS OF ALEXANDRIA

J. D. SWIFT, University of California at Los Angeles and Institute for Advanced Study

1. Introduction. The name of Diophantus of Alexandria is immortalized in the designation of indeterminate equations and the theory of approximation. As is perhaps more often the rule than the exception in such cases, the attribution of the name may readily be questioned. Diophantus certainly did not invent indeterminate equations. Pythagoras was credited with the solution $(2n+1, 2n^2+2n, 2n^2+2n+1)$ of the equation $x^2+y^2=z^2$; the famous Cattle Problem of Archimedes is far more difficult than anything in Diophantus, and a large number of other ancient indeterminate problems are known. Further, Diophantus did not even consider the most common type of problem called by his name, the linear equation or system of equations to be solved in integers.

Nevertheless, on at least three grounds the place of Diophantus in the development of mathematics is secure. On all the available data he was the first to introduce systematic algebraic procedures to the solution of non-linear indeterminate equations and the first to introduce extensive and consistent algebraic notation representing a tremendous improvement over the purely verbal styles of his predecessors (and many successors). Finally, the rediscovery of the book through Byzantine sources greatly aided the renaissance of mathematics in western Europe and stimulated many mathematicians, of whom the greatest was Fermat. (Much of Fermat's work is known from notes written in his copy of Diophantus.) [1]

Of Diophantus as an individual we have essentially no information. A famous problem in the Greek Anthology indicates that he died at the age of 84, but in what year or even in which century we have no definite knowledge. He quotes Hypsicles and is quoted by Theon, the father of Hypatia. Now Hypsicles, in the introduction to his book, the so-called Book XIV of Euclid, places himself within a generation or so of Apollonius of Perga whose time is definitely established by the rulers to whom he dedicates his works. Thus we may put Hypsicles in the early or middle part of the second century B.C. with reasonable accuracy [17]. Theon, on the other hand, definitely saw the eclipse of 364 A.D. [10]. Within this gap of five hundred years, historians are at liberty to place Diophantus wherever he best fits their theories of historical development [10, 14]. The majority follow [2] and, on the basis of a dubious reference by the Byzantine Psellus (C.1050), assign him to the third century A.D.

2. The Arithmetic. The surviving work of Diophantus consists of six books (sometimes divided into seven) of the *Arithmetic* and a fragment of a work on polygonal numbers. The introduction to the *Arithmetic* promises thirteen books. The position and content of the missing six or seven books is a matter of conjecture. (The reader is reminded that a "book" is a single scroll and represents the material contained in twenty to fifty pages of ordinary type.)

These books may be summarized as follows: Book I—Determinate systems

of equations involving linear or quadratic methods. Books II–V—Equations and systems of equations, the majority of which are quadratic indeterminate although Books IV and V contain a selection of cubic equations, determinate and indeterminate. Book VI—Equations involving right triangles. All books consist of individual problems and their solutions in positive rationals. In the ordering of the problems some consideration has been given to relative difficulty and interrelation of material, but the over-all impression is of a disconnected assortment.

3. Notation. The numerical notation used by Diophantus is, of course, the standard Hellenistic notation which uses the letters of the Greek alphabet with three archaic letters added to give 27 different symbols; [6, 7] the first nine stand for units, the second for tens and the last for hundreds. Thus, for any further notation, either non-alphabetic symbols or monogrammatic characters were required.

There is a single symbol for the unknown quantity. This may be a monogram for αριθμος. The symbol for "minus" is apparently a monogram for the root of λειψις [5]. Addition is indicated by juxtaposition. The powers of the unknown are designated by easily recognizable monograms, the square by Δ^{υ} for δυναμις, the cube by K^{υ} for κυβος. Higher powers are formed from these by addition, *i.e.*, the fifth power is considered as square-cube. To avoid ambiguity it is necessary to have a special symbol for the zero-order terms also, a monogram for μοναδος, and to write all the negative terms together. Thus, if we adopt an equivalent set of conventions in English, retaining Arabic numerals and the letter x for the indeterminate, the expression: $6x^4+23x^3-2x^2+x-5$ would appear as $SS^{\upsilon}6C^{\upsilon}23X1MS^{\upsilon}2U^{\upsilon}5$.

Fractions were represented either in the inverse position to the present day or by inserting the word for "divided by" between the numerical expressions on the same line. Reciprocals of integers and negative powers of the unknown are designated by a special symbol placed after the number or power.

The most important limitation of this notation is the restriction to one unknown. Since practically all the problems require the determination of several quantities, a considerable part of Diophantus' work lies in the reduction to a single quantity. Further, no general solution in expressed parameters is possible. Even if a general method is indicated, it must be restricted in its presentation to a specific numerical case.

A particular problem will illustrate the situation. In problem 1, Book IV, it is desired, in modern terms, to solve the system: $x^3+y^3=a$, $x+y=b$. Essentially the method is to let $x=z+b/2$, $y=b/2-z$. Substitution in the first equation now yields a binomial quadratic. Let us look at this problem in a translation as bald as possible:

To partition a given number into two cubes of which the sum of the sides is given: Let the number to be partitioned be 370 and the sum of the sides $U^{\upsilon}10$. Let the side of the first cube be $x1U^{\upsilon}5$, the latter term of which is half the sum

of the sides. Therefore, subtracting, the side of the other cube is $U^{\alpha}5Mx$. Then the sum of the cubes will be $S^{\alpha}30U^{\alpha}250$. This is equal to $U^{\alpha}370$ as is given and x becomes $U^{\alpha}2$. As to the original numbers, the first side will be $U^{\alpha}7$ and the second, $U^{\alpha}3$. The first cube, 343; the second, 27.

4. Diophantine algebra. With this problem in mind let us turn to some aspects of Greek and Babylonian mathematics. A number of tablets [15, 16], both old Babylonian (1800–1600 B.C.) and Seleucid (300 B.C. and later), exist which teach the solution of equations which can be reduced to the forms $x+y=a$; $xy=b$ [10, 12, 13]. Again Euclid's Elements II, 5, 6 can best be viewed as giving solutions to these problems [12]. In modern notation, the procedure in both cases is to write $x=a/2+z$, $y=\pm(a/2-z)$; $xy=b=\pm(a^2/4-z^2)$; $z=\sqrt{a^2/4\pm b}$. Now Diophantus in I (27, 30) considers the same equations, solves them the same way and applies the basic idea repeatedly as in the quoted problem. Other examples can be followed in a similar way, *e.g.*, $x^2+y^2=a$, $xy=b$. (See [13] for a complete discussion of quadratic equations in antiquity.)

Let us now compare the treatment in the three cases: The tablets consist of lists of problems of varying complexity each framed in specific numbers and quantities. The problems are not "practical" nor in any sense rigorously geometric; men are added to days, lengths to areas, areas are multiplied, *etc.* It is clear that the basic thought is purely algebraic. The problems are so set that the solutions are positive integers or terminating sexagesimal fractions such that the roots can be obtained from tables of squares, but from other tablets we learn of approximations to non-terminating rationals like 1/7 or irrationals like $\sqrt{2}$.

The Euclidean problems are cast in the form of propositions about line segments, squares and rectangles. Their generalizations in II, 28, 29, concern parallelograms. The propositions are general and the result is deduced rigorously from the postulational basis. The results are line segments which may well be incommensurable with the original segments; *i.e.*, "irrational" answers are acceptable.

In Diophantus the problems are formulated in terms of abstract numbers but a "number" is always positive rational. The solutions are worked out in terms of particular numerical examples. This procedure may be considered analogous to carrying out a geometrical construction in terms of particular line segments and, indeed, Diophantus probably intended that his problems should be read in this manner. There is, however, no pretense at postulational development. No general propositions are stated even where the solution implies them. Restrictions on the choice of initial values are not always given; in the case of I, 27, we are informed that $a^2/4-b$ must be a square but in the problem in the previous section no restriction is mentioned. The most reasonable conclusion is that he did not know the form of the restriction or did not know how to express numbers that satisfied the restriction. The authors of [5] and [4] disagree but on the naive ground that since he did come up with workable numbers 370 and 10, he must have had some way of generating them. The answer is obvious; he

generated them from the answer.

Like the Babylonians, Diophantus had no qualms about adding areas and lengths (See VI, 19 in 5) although, to be precise, he says that he adds "the number in the area" to "the number in the length." His algebraic technique is tremendously advanced beyond anything we possess of the Babylonians. The complicated cubic and higher degree equations and the indefinite equations are not even suggested in Babylonian algebra. The latter had examples of binary cubics and a few other higher degree equations soluble by tables; they also knew general forms for Pythagorean numbers and obtained solutions of $x^2-2y^2=1$ but this is as far as our present evidence takes them. Even in the quadratic case there may be a difference [13]. When a quadratic is to be solved, Diophantus makes some effort to choose the variable so that a binomial equation results, but if this is not practicable, the general quadratic formula (positive sign before the radical) is used without further comment. The question is still at issue whether the Babylonians ever solve a quadratic without bringing it into some normal form involving a known sum or difference and product.

It is useless even to try to guess what proportion of the advanced problems and methods are Diophantus' own. Most modern historians postulate a continuous underlying tradition of oriental algebraic methods in Greek mathematics rather than a sudden invasion in the Roman period. If this be so, texts and problem lists would certainly have existed. It is probable that the *Arithmetic* was in good part a compilation of such a quality that the predecessors were no longer held in repute. There are traces of the Diophantine notation elsewhere; Heron (60 A.D.) used the same minus sign for example, but no evidence exists that the semi-algebraic notation or the general methods it permitted were used before the publication of the *Arithmetic*.

To sum up, the basic algebraic approach in Diophantus is Babylonian. The generality and abstraction is Greek. The work may be viewed as an episode in the decline of Greek mathematics [12] or as the finest flowering of Babylonian algebra [10].

5. Indeterminate problems. In giving translations of several illustrative problems, I have avoided the usual practice of direct translation. Instead, I adhere carefully to the method of the original while replacing the particular numbers used by parameters. The rationale may thus be conveyed with less verbal explanation than if the presentation were given in its original special form. At the same time, the full power of the method is apparent.

II. 9: If $n=a^2+b^2$, find other representations of n as the sum of two squares: Modify a to $x+a$. The corresponding modification of b may be written $(rx-b)$. Here x is the unknown, a, b and r were assigned specific values.

$$n = a^2 + b^2 = (x + a)^2 + (rx - b)^2$$
$$= (r^2 + 1)x^2 + (2a - 2br)x + a^2 + b^2.$$

Thus $x=(2br-2a)/(r^2+1)$ where r may be any rational such that the re-

quired quantities are positive.

Note the clever choice of the unknown; fixing x and solving for r would leave a condition on x still to be met; b^2-x^2-2ax would have to be made a square. Again, if a is increased by a fixed amount and the unknown is taken as the corresponding decrease in b, the result does not come out at once. The choice of $rx-b$ instead of $b-rx$ was dictated solely by the numerical values selected which happened to make $b-rx$ negative. (But see V, 24, below.) Euler wrote $a^2+b^2=(a+rx)^2+(b-qy)^2$ which results in a more symmetric solution but this concept is foreign to Diophantus' notation and the solution above is quite general.

Here would be a perfect opportunity to state a proposition instead of a problem. A proof of the theorem: "Any number which is the sum of two squares can be represented as such in an infinite number of ways," is contained in the solution above.

III. 6: Find three numbers whose sum is a square and such that the sum of any two is a square:

$$x+y+z=t^2$$
$$x+y=u^2$$
$$y+z=v^2$$
$$x+z=w^2.$$

Here Diophantus assigns a definite value to w, or, in modern notation, lets it play the role of the parameter. He then chooses an unknown, r, restricting it as follows: Let $t=r+1$, $u=r$, $v=r-1$. Then $z=2r+1$, $y=r^2-4r$, $x=4r$ and $w^2=6r+1$. Thus $r=(w^2-1)/6$ where w is an arbitrary rational exceeding 5 (so that y is positive). So $x=(2w^2-2)/3$; $y=(w^2-1)(w^2-25)/36$; $z=(w^2+2)/3$.

This problem was chosen to illustrate two points. First Diophantus is not interested in generality except as an incidental by-product. A considerable increase in generality can be obtained merely by replacing $r\pm1$ by $r\pm s$ in the solution and this possibility could easily have been indicated by the addition of a single phrase. Second, the choice of wording of the problems is often peculiar from a modern viewpoint. This problem is clearly equivalent to the single equation: $u^2+v^2+w^2=2t^2$.

Incidentally, using methods available to Diophantus but probably exceeding his control of notation, a much more general solution of this equation is available than the system $(u, v, w, t)=((w^2-1)/6, (w^2-7)/6, w, (w^2+1)/6)$ given above. The equation being homogeneous, it will be more convenient to solve in integers. Let $w=rs$, $u=s^2-p$, $v=s^2-q$, then

$$s^4+(r^2/2-p-q)s^2+(p^2+q^2)/2=t^2.$$

The left hand side is a perfect square if $p^2+q^2=2k^2$, $r^2/2-p-q=2k$. The first of these is the problem of finding three squares in arithmetic progression. It does not occur specifically in the *Arithmetic*, probably because it is too simple in the

rational case, reducing essentially to $a=b=1$ in the problem above. It will be more convenient to take a solution derived from the solution given to the Pythagorean equation by Euclid. If $X^2+Y^2=Z^2$, $(X+Y)^2+(X-Y)^2=2Z^2$. Thus $p=-m^2+2mn+n^2$, $q=m^2+2mn-n^2$, $k=m^2+n^2$; $r^2=4(m+n)^2$ and $r=2(m+n)$. Thus $(u, v, w, t)=(s^2+m^2-2mn-n^2, s^2-m^2-2mn+n^2, 2(m+n)s, s^2+m^2+n^2)$. The previous solution is obtained by setting $m=2$, $n=1$ and dividing by 6.

V. 24: Find a solution of $x^4+y^4+z^4=t^2$.

If $t^2=(x^2-m)^2$, $x^2=(m^2-y^4-z^4)/2m$. Thus an integer m must be found so that $(m^2-y^4-z^4)/2m$ is a square. Let $m=y^2+z^2$ so $x^2=y^2z^2/(y^2+z^2)$. Thus y^2+z^2 must be a square, say $(y+r)^2$. Then $y=(z^2-r^2)/2r$. Thus

$$(x, y, z, t)=\frac{z^3-r^2z}{z^2+r^2}, \quad \frac{z^2-r^2}{2r}, \quad z, \quad \frac{z^8+14z^4r^4+r^8}{4r^2(z^2+r^2)^2}.$$

This example has been chosen for three reasons. First, it is of great historical interest. To this problem Fermat appended a note: "Why does Diophantus not ask for the sum of *two* biquadrates to be a square? This is, indeed, impossible . . . ." Later, Euler conjectured that it was also impossible to find three fourth powers whose sum was a fourth power; *i.e.*, to replace t^2 by t^4. This question remains unsolved.

Second, the problem indicates what happens when the notation is insufficient. First, the chosen unknown is x; m, y, z are assigned specific values to indicate that they play the role of parameters. But the problem cannot be completed, so the author turns to a sub-problem in which y is the unknown.

Finally, the problem contains a curious case of indifference to sign. The quantity x^2-m is, in fact, the negative root of $t^2=(x^2-m)^2$. Since only the square is used, no harm is done but we must remember that, to Diophantus, the quantity x^2-m, which he used, did not exist. The reader may find it interesting to see why x^2+m was not used by trying it. Why $m-x^2$ which is positive and produces the same result was not preferred is a matter for conjecture.

VI. 19: Find a right triangle such that its area added to one of its legs is a square while the perimeter is a cube.

First form the triangle $(2x+1, 2x^2+2x, 2x^2+2x+1)$. The perimeter is $4x^2+6x+2=(4x+2)(x+1)$. Since it is difficult to make a quadratic a cube, consider in turn the triangle $(2x+1)/(x+1)$, $2x$, $2x+1/(x+1)$ obtained by dividing through by $x+1$. The perimeter is $4x+2$ and the area is $(2x^2+x)/(x+1)$. Adding $(2x+1)/(x+1)$ to the latter we have $2x+1$. Thus $4x+2$ is required to be a cube and $2x+1$ a square. The obvious value for $2x+1$ is 4. Thus $x=3/2$ and the triangle is 8/5, 3, 17/5.

It is not clear whether or not Diophantus implies the more general solution $2x+1=4r^6$, $x=(4r^6-1)/2$; probably not.

This problem is illustrative of the rather peculiar problems considered throughout Book VI and of the complete freedom from geometrical considera-

tions. To Euclid such phrases as "the sum of one side and the area" would have been shocking nonsense.

6. An approximation problem. In V. 9 it is required as a sub-problem to find two squares, both exceeding 6, whose sum is 13. Since we have, in the first example of the preceding section, a general method of partitioning a number into two squares when one such partition is given, it is merely necessary to set the two values equal, solve for the parameter and approximate this solution in rationals. If this is done with $a=2$ and $b=3$, we find $r=5+\sqrt{26}$. Approximating by $r=10$, $13=(258/101)^2+(257/101)^2$ and it is readily seen that the conditions are met.

Of course, Diophantus could not do this since the parameters were not expressed. He first finds a number slightly greater than $\sqrt{13/2}$. $13/2=26/4$; if $\sqrt{26}<5+1/x$, $x^2<10x+1$; let $x=10$, then $\sqrt{13/2}\sim 51/20$. Now $51/20=3-9/20=2+11/20$. Thus we wish to find a number near $1/20$ such that $(3-9y)^2+(2+11y)^2=13$. Then $y=5/101$ and the squares are precisely those obtained above.

The problem is typical of the approximate methods used. To approximate the nth root of a rational, first write it in the form p/q^n by multiplying numerator and denominator by the necessary integer to make the denominator a perfect nth power. Then multiply p by the nth powers of successive integers until pa^n is sufficiently close to a perfect nth power, say b^n. The approximation is then b/aq. To improve an approximation a_1 to $\sqrt{a}$, set $(a_1+1/x)^2=a$ and approximate x.

7. Transmission of Diophantus. When the Arabs overran the Southeastern Mediterranean in the 7th century, they came into possession of manuscripts of works which had been published in sufficiently large editions to survive the wars attendant on the breakup of the Roman Empire and the lack of interest in learning of the early Christians. Among these was the *Arithmetic* or at least a portion of it. Translations and commentaries were published in Arabic. These have all been lost; their only trace is in bibliographers' references. When the Arabs formulated their own algebra, they apparently appealed directly to the basic Oriental tradition previously cited. The beginnings of an algebraic notation and the abstract numbers are nowhere to be seen. With the sole exception of the problems mentioned in section 3 as common to the whole ancient world (Diophantus—I, 27–30) not one problem from the *Arithmetic* is found in the algebra of Al-Khwarizmi or, as far as is known, in any other basic Oriental text [13]. Probably the Arabs found Diophantus too impractical for their utilitarian mathematics and the Hindus, if they ever saw the *Arithmetic*, were interested in other problems such as the theory of linear indeterminate problems.

In the other reservoir of learning, Byzantium, the manuscripts of Diophantus lay almost unnoticed for eight centuries. We do not know when the missing books were lost but the part which we now possess escaped the sack of Constantinople by the Crusaders in 1204 and later in the same century M. Planudes and G. Pachymeres wrote commentaries on the first part of the *Arithmetic*. At

some time, probably in the course of the emigration of the Byzantine scholars during the Turkish conquests, copies were brought to Italy and Regiomontanus saw one there between 1461 and 1464.

The first translation to Latin was made by W. Holzmann who wrote under the Greek version of his name, Xylander. This translation was published in 1575. Meanwhile, Bombelli, in 1572, distributed all the problems in the first four books among problems of his own in a text on algebra. Bachet, borrowing liberally from Bombelli and Holzmann made another translation in 1621 and a second edition was published in 1670 including Fermat's marginal notes. In the next two centuries various translations were made into modern languages which were based primarily on the editions just mentioned by Holzmann and Bachet. Finally in 1890, P. Tannery prepared a definitive edition of the Greek text with a translation into simple mathematical Latin using modern numerical and algebraic notation. From this work the three excellent translations listed in the bibliography have been prepared. The references to the last two paragraphs are the commentaries in [2] and [6], particularly the latter which has been followed rather closely.

Bibliography

A. Greek Text with Latin Translation:

[1] C. G. Bachet: Diophanti Alexandrini Arithmeticorum libri sex, *etc.* Paris, second edition, 1670.

[2] P. Tannery: Diophanti Alexandrini opera omnia cum Graecis commentariis, Teubner, vol. 1, 1893, vol. ii, 1895.

B. Modern translations based on [2]:

[3] A. Czwalina: Arithmetik des Diophantos aus Alexandria, Göttingen, Vandenhoek, 1952.

[4] P. ver Eeke: Diophante d' Alexandrie. Les six livres arithmetiques et le livre des nombres polygones. Bruges, Descée, 1926.

[5] T. L. Heath: Diophantus of Alexandria, Cambridge, 1910.

C. Histories and Compilations of Ancient Mathematics:

[6] T. L. Heath: History of Greek Mathematics (Two volumes), Oxford, 1921.

[7] ——— A Manual of Greek Mathematics, Oxford, 1931.

[8] G. Loria: Le Scienze esatte nell' antica Grecia, 2nd edition, Milan, Hoepli, 1914.

[9] G. H. F. Nesslemann: Die Algebra der Griechen, Reimer, Berlin, 1842.

[10] O. Neugebauer: The Exact Sciences in Antiquity, Princeton, 1952.

[11] I. Thomas: Selections Illustrating the History of Greek Mathematics, Harvard and Cambridge (Loeb Classics) 1939.

[12] B. L. van der Waerden: Science Awakening, P. Noordhoff, Groningen, 1954 (English translation by A. Dresden).

D. Miscellaneous References:

[13] S. Gandz: The Origin and Development of the Quadratic Equations in Babylonian, Greek and Early Arabic Algebra, Osiris, vol. 3 (1938) pp. 405–557.

[14] J. Klein: Die griechische Logistik und die Entstehung der Algebra, Quellen and Studien zur Ges. der Math. Abt. B, vol. 3, 1934–6, pp. 18–105; 122–235.

[15] O. Neugebauer: Mathematische Keilschrift-Texte, 3 Vols., Springer, 1935–7 = Quellen und Studien zur Ges. der Math. Abt. A, vol. 2.

[16] ——— and A. Sachs: Mathematical Cuneiform Texts, American Oriental Society, New Haven, 1945 = Am. Oriental Series, vol. 29.

[17] Paulys Real Encyclopädie der Classischen Altertums Wissenschaft, Stuttgart, 1893.

ON THE ZEROS OF A CUBIC RECURRENCE

M. F. SMILEY, The State University of Iowa

A cubic recurrence $(T)=(T_1, T_2, T_3, \cdots)$ is defined by real numbers T_0, T_1, T_2, and

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, 2, \cdots),$$

where P, Q, R are real numbers. We shall assume that the roots u, v, w of $f(z)=0$, where

$$f(z) = z^3 - Pz^2 + Qz - R$$

is the *companion* of (T), are nonzero, real, and have distinct absolute values. If $T_k=0$, we call k a *zero* of (T). If $r\geqq 0$ is fixed and (T') is defined by $T_n'=T_{n+r}$, we call (T') a *translate* of (T). Every translate of (T) has the same companion as (T). We shall also assume that $(T)\neq(0, 0, 0, \cdots)$ which holds (1) if and only if no three consecutive T_n are zero, or (2) if and only if it holds for every translate of (T). Then we may show that there are real numbers $(U, V, W)\neq(0, 0, 0)$ and such that

$$T_n = Uu^n + Vv^n + Ww^n \qquad (n = 0, 1, 2, \cdots).$$

THEOREM. *The cubic recurrence (T) has no more than three zeros.*

Morgan Ward [1] recently proved this result under the restriction that u, v, w were integers co-prime in pairs. Our proof is based in part on the ideas of Ward, but is more direct; the essential tool is the familiar Descartes' rule of signs.

We begin with the following lemma.

LEMMA 1. *If $T_0=T_r=T_s=0$ with $s>r>0$, then $x=u$, v, w are roots of*

$$g(x) = (w^r - u^r)x^s - (w^s - u^s)x^r + u^rw^r(w^{s-r} - u^{s-r}) = 0. \tag{1}$$

Proof. The determinant of the matrix

$$C = \begin{bmatrix} 1 & u^r & u^s \\ 1 & v^r & v^s \\ 1 & w^r & w^s \end{bmatrix}$$

is zero, since, by hypothesis, $[U, V, W]C=[0, 0, 0]$. The lemma follows on suitable expansion of this determinant.

LEMMA 2. *If $T_0=T_r=T_s=0$ with $s>r>0$, then $d=(r, s)$ is odd.*

Proof. First note that u, v, w cannot all be positive. For Descartes' rule permits $g(x)=0$ at most two positive real roots. Now suppose that d is even. With

$u'=u^d$, $v'=v^d$, $w'=w^d$, $x'=x^d$, $r'=r/d$, $s'=s/d$, we find that $x'=u'$, v', w' are distinct positive real roots of the equation obtained by priming all of the letters of (1).

Let w be the root of $f(z)=0$ of greatest absolute value. Since replacing T_n by $(-1)^nT_n$ changes the signs of the roots of the companion without changing the zeros, we may assume that $w>0$. (See [1]).

LEMMA 3. *Let* $T_0=T_r=T_s=0$ *with* $s>r>0$. *Then*

(1) *if* r *and* s *are both odd,* $f(z)=0$ *has exactly one positive root,*

(2) *if* r *is even and* s *is odd, two roots of* $f(z)=0$ *are positive,*

(3) r *odd and* s *even is not possible.*

Proof. We have seen that not all the roots of $f(z)=0$ can be positive. We may assume that $u<0$ so that $-w<u<0$. The signs of the coefficients of $g(x)$ in the three cases are: (1) $+--$, (2) $+-+$, (3) $+--$, while the signs of the coefficients of $g(-x)$ are (1) $-+-$, (2) $--+$, (3) $++-$. Descartes' rule and Lemma 1 now yield the stated conclusions.

Proof of the theorem. We see that (T) cannot have three even (or three odd) zeros since a suitable translate of (T) would then violate Lemma 2. Thus (T) has at most four zeros and no odd (even) zero is between two even (odd) zeros or a suitable translate of (T) would violate case (3) of Lemma 3. There remains the possibility that (T) has four zeros k_1, k_2, k_3, k_4 with $0\leqq k_1<k_2<k_3<k_4$ and $k_1\equiv k_2$ and $k_3\equiv k_4\not\equiv k_1$ modulo 2. This occurs if and only if some translate of (T) has four zeros 0, r, s, t with $0<r<s<t$ and with r even, s and t odd. But then Lemma 3 applied to $T_0=T_s=T_t=0$ and to $T_0=T_r=T_s=0$ yields a contradiction: $f(z)=0$ has exactly one positive root and $f(z)=0$ has two positive roots. The proof is complete.

Concluding remark. For non-degenerate real recurrences (T) of order n, the analogue of Lemma 2 (with an analogous proof) limits the number of zeros of (T) to $2n-2$. No doubt there is an analogue of Lemma 3 which will reduce this crude bound.

Reference

1. Morgan Ward, On the number of vanishing terms in an integral cubic recurrence, this MONTHLY, vol. 62, 1955, pp. 155–160.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

AN EIGENVALUE INEQUALITY FOR THE PRODUCT OF NORMAL MATRICES

M. MARCUS, University of British Columbia

Let s_{ij} be n^2 real numbers satisfying

$$s_{ij} \geqq 0; \qquad i,j = 1, \cdots, n \tag{1}$$

$$\sum_{k=1}^{n} s_{kj} = 1; \qquad j = 1, \cdots, n \tag{2}$$

$$\sum_{k=1}^{n} s_{ik} = 1; \qquad i = 1, \cdots, n. \tag{3}$$

Let ϕ_{ij} be any n^2 real numbers and set

$$f(s) = \sum_{i,j=1}^{n} \phi_{ij} s_{ij}.$$

Let p denote any permutation of the numbers $(1, 2, \cdots, n)$. It is known [1] that

$$\min_{p} \sum_{i=1}^{n} \phi_{ip(i)} \leqq f(s) \leqq \max_{p} \sum_{i=1}^{n} \phi_{ip(i)}. \tag{4}$$

The inequality (4) may be directly applied to obtain the following

THEOREM. *If A and B are n-square normal matrices with eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$, $i=1, \cdots, n$, then*

$$\min_{p} \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B) \leqq \operatorname{Re} \sum_{i=1}^{n} \lambda_i(AB) \leqq \max_{p} \operatorname{Re} \sum_{i=1}^{n} \lambda_i(A)\lambda_{p(i)}(B). \tag{5}$$

Proof. First note that

$$2 \operatorname{Re} \sum_{i=1}^{n} \lambda_i(AB) = \operatorname{tr}(AB + B^*A^*).$$

Let U and V be unitary matrices such that

$$U^*AU = \operatorname{diag}(\lambda_1(A), \cdots, \lambda_n(A)) = \operatorname{diag}(\lambda(A))$$

and

$$V^*BV = \operatorname{diag}(\lambda_1(B), \cdots, \lambda_n(B)) = \operatorname{diag}(\lambda(B)).$$

Then

$$\begin{aligned}\operatorname{tr}(AB + B^*A^*) &= \operatorname{tr}(U \operatorname{diag}(\lambda(A))U^*B + B^*U \operatorname{diag}(\bar{\lambda}(A))U^*) \\ &= \operatorname{tr}(U\{\operatorname{diag}(\lambda(A))(U^*BU) + (U^*B^*U)\operatorname{diag}(\bar{\lambda}(A))\}U^*) \\ &= \operatorname{tr}(\operatorname{diag}(\lambda(A))U^*V \operatorname{diag}(\lambda(B))V^*U \\ &\quad + U^*V \operatorname{diag}(\bar{\lambda}(B))V^*U \operatorname{diag}(\bar{\lambda}(A))).\end{aligned}$$

Let $T = V^*U$, $T^* = U^*V$ and then

$$\begin{aligned}(T^* \operatorname{diag}(\lambda(B))T)_{ii} &= \sum_{k,a=1}^{n} T^*_{ik}(\operatorname{diag}(\lambda(B)))_{ka}T_{ai} \\ &= \sum_{k=1}^{n} T^*_{ik}\lambda_k(B)T_{ki} \\ &= \sum_{k=1}^{n} |T_{ik}|^2\lambda_k(B).\end{aligned}$$

We set $|T_{ik}|^2 = s_{ik}$ and note that since T is unitary (1), (2) and (3) are satisfied. Proceeding, we have

$$\begin{aligned}\operatorname{tr}(AB + B^*A^*) &= \sum_{i=1}^{n} \lambda_i(A)(T^* \operatorname{diag}(\lambda(B))T)_{ii} + \sum_{i=1}^{n} \bar{\lambda}_i(A)(T^* \operatorname{diag}(\bar{\lambda}(B))T)_{ii} \\ &= \sum_{i=1}^{n} \lambda_i(A) \sum_{k=1}^{n} \lambda_k(B)s_{ik} + \sum_{i=1}^{n} \bar{\lambda}_i(A) \sum_{k=1}^{n} \bar{\lambda}(B)s_{ik} \\ &= \sum_{i,k=1}^{n} (\lambda_i(A)\lambda_k(B) + \bar{\lambda}_i(A)\bar{\lambda}_k(B))s_{ik}.\end{aligned}$$

If we set $\phi_{ik} = \lambda_i(A)\lambda_k(B) + \bar{\lambda}_i(A)\bar{\lambda}_k(B)$ it is clear that (4) applies and we obtain the inequality (5).

Reference

1. G. Birkhoff, Three observations on linear algebra, Universidad Nacional de Tucuman, Revista, Serie A, vol. 5, 1946, pp. 147–151.

SEQUENCES OF 0's AND 1's AND TOEPLITZ METHODS OF SUMMABILITY

G. M. Petersen, University of Oklahoma

It has been shown by Schur [2] that no regular matrix summability method sums all bounded sequences. In fact, no regular summability method sums all the sequences consisting of 0's and 1's [1], [3]. On the other hand, there exist many methods which sum all (that is, both) sequences of 0's and 1's in which no pairs of consecutive 0's or consecutive 1's occur. This suggests that we consider the next simplest case of sequences of 0's and 1's containing no triplet of consecutive 0's or of consecutive 1's.

THEOREM. *For each regular Toeplitz matrix* $A=(a_{mn})$, *there exists a sequence* $\{s_n\}$ *of* 0*'s and* 1*'s, containing no triplet of consecutive* 0*'s or of consecutive* 1*'s, and such that* $\{s_n\}$ *is not summable by* A.

Proof. Since every regular matrix is equivalent, for bounded sequences, to a matrix such that, for every m, there are i_m and j_m, $i_m<j_m$, with $a_{mn}=0$ if $n<i_m$ or $n>j_m$ and $\lim_{m\to\infty} i_m=\infty$, we may suppose that A has this property. Moreover, by deleting a set of rows from A we obtain a matrix $B=(b_{mn})$ such that the above $\{i_m\}$ and $\{j_m\}$ may be taken to be multiples of 3 and $i_1<j_1<i_2<j_2<\cdots<i_k<j_k<\cdots$. Since the B transform of every sequence is a subsequence of its A transform, it is evident that every sequence which is A summable is also B summable, so that it suffices to consider the matrix B.

For $k=4p$, $p=1, 2, \cdots$ and for $i_k\leqq n<j_k$, we define s_n by repetitions of the triplet (1, 0, 0). In the same way, for $k=4p+1$, $k=4p+2$, $k=4p+3$ respectively and $i_k\leqq n<j_k$, we define s_n by repetitions of the triplets (0, 1, 0), (0, 0, 1) and (1, 1, 0). The remaining entries in $\{s_n\}$ may be made so that no consecutive triplet of 0's or of 1's appears.

We assume that B sums $\sigma=\{s_n\}$ as well as the four sequences

$$\sigma_1=\{1,0,0,1,0,0,\cdots\},\qquad \sigma_2=\{0,1,0,0,1,0\cdots\},$$
$$\sigma_3=\{0,0,1,0,0,1,\cdots\},\quad\text{and}\quad \sigma_4=\{1,1,0,1,1,0,\cdots\},$$

for otherwise the theorem would be proved. It now follows that all five sequences are summed by B to the same value $c=c_1=c_2=c_3=c_4$. But it is evident that $c_1+c_2=c_4$ and $c_3+c_4=1$, so that $2c=c$ and $2c=1$. This contradiction establishes the theorem.

Hill [1] has shown that the set of sequences of 0's and 1's summable by any regular method, when considered as a set of real numbers in binary expansion, is of the first category. Since the set of 0's and 1's of our theorem is of the first category, Hill's result can not be applied.

References

1. J. D. Hill, The Borel property of summability methods, Pac. Jour. of Math., vol. 1, no. 3, 1951, pp. 399–409.

2. I. Schur, Über lineare Transformationen in der Theorie der unendlichen Reihen, J. Reine Angew. Math., vol. 151, 1921, pp. 79–111.

3. H. Steinhaus, Some remarks on the generalization of the notion of limit (in Polish), Prace Matematyczno Fizyczne, vol. 22, 1921, pp. 121–134.

ON COMPLEMENTARY GRAPHS

E. A. NORDHAUS and J. W. GADDUM, Michigan State University

1. Introduction. We are concerned in this note with proving certain relations concerning the chromatic numbers of complementary linear graphs. As the terminology suggests, the discussion is motivated by the map-coloring problem. A *graph*, for our purposes, is a finite set of abstract elements, called nodes, and certain unordered pairs of distinct nodes, called edges. The number of nodes is

called the *order* of the graph. A *complete* graph on n nodes is the set of all $n(n-1)/2$ edges and the n nodes determining these edges. Two graphs G and G' having the same n nodes are called *complementary* if each edge determined by the n nodes is in either G or G' but not in both.

A *k-coloring* of a graph G is an assignment to each node of one of the numbers $1, 2, \cdots, k$ in such a way that if the edge (a, b) is in G, the nodes a and b are assigned different numbers. The smallest k for which a graph G admits a k-coloring is called the *chromatic number* of G. Clearly if G has n nodes, $1 \leqq k \leqq n$. We denote the chromatic number of the complementary graph G' by k'.

2. Bounds. Our principal result is the following: If G and G' are complementary graphs on n nodes whose chromatic numbers are k, k' respectively, then

$$2\sqrt{n} \leqq k + k' \leqq n + 1 \tag{1}$$

$$n \leqq kk' \leqq \left(\frac{n+1}{2}\right)^2, \tag{2}$$

and these are the best possible bounds, in the sense that each bound is assumed for infinitely many values of n. The lower bound in (2) has been established by Zykov [1].

Proof. Consider a k-coloring of G, and let n_i be the number of nodes assigned the i-th color, for $i=1, 2, \cdots, k$. Then $n_1+n_2+\cdots+n_k=n$ and $\max n_i \geqq n/k$. Now if two nodes are assigned the same color in coloring G, the edge they determine must be in G', and in coloring G' they must be assigned different colors. Therefore $k' \geqq \max n_i \geqq n/k$, or $kk' \geqq n$. Then $(k-k')^2 \geqq 0$ implies $(k+k')^2 \geqq 4kk'$ and $k+k' \geqq 2\sqrt{n}$. This establishes the lower bounds in (1) and (2).

To show that $k+k' \leqq n+1$, we use induction on n. The result is evident for $n=1$, and we assume that $k+k' \leqq n+1$ holds for complementary graphs on n nodes. To the sum of G and G', which consists of the complete graph on these n nodes, we adjoin a node P. Of the n edges emanating from P, let q be adjoined to G and the remaining $n-q$ to G'. If H and H' are the graphs so determined (each of order $n+1$), and these have chromatic numbers l and l', respectively, then

$$l \leqq k + 1, \qquad l' \leqq k' + 1.$$

These inequalities are evident from the fact that the addition of a node to a graph and some or all edges from this node to nodes of the graph can increase the chromatic number by at most one. Then $l+l' \leqq n+2$ except possibly when $l=k+1$, $l'=k'+1$. In this event, the graph H has the property that if node P and all edges of H emanating from P are deleted, the chromatic number is decreased. Such a point is termed a *critical point* of the graph, and a simple argument shows that $q \geqq k$. Similarly, $n-q \geqq k'$ and thus $k+k' \leqq n$, from which $l+l' \leqq n+2$. This completes the induction. Finally, the inequality $4kk' \leqq (k+k')^2$ implies that $kk' \leqq ((n+1)/2)^2$, establishing the remaining upper bound.

The following three examples show that each of the bounds established is

actually assumed for infinitely many values of n. Example 1. Let G be a complete graph on n nodes. Then $k=n$, $k'=1$, so $k+k'=n+1$, $kk'=n$. Example 2. Let $n=q^2$, and consider q groups of q nodes each, where the nodes in each group bear the same color, q colors being used. The edges of G are then defined as those edges which join nodes of different colors. Then $k=q$, $k'=q$, so $k+k'=2q=2\sqrt{n}$, $kk'=q^2=n$. Example 3. Let $n=2t-1$, and separate the nodes of G into two groups containing t and $t-1$ nodes, respectively. The edges of G are those of the complete graph of order t determined by the first group of t nodes. Then $k=t$, $k'=(t-1)+1=t$, $k+k'=n+1$, and $kk'=t^2=((n+1)/2)^2$.

No graph with $n>1$ has $k+k'=2\sqrt{n}$ and $kk'=((n+1)/2)^2$ holding simultaneously. A simple geometric argument shows that when either of the two equalities just mentioned occurs, then $k=k'$ must hold.

Reference

1. A. A. Zykov, On some properties of linear complexes, Math. Sbornik. NS 24(66), 163–188 (1949). [American Math. Society Translation No. 79].

ON THE PROPAGATION OF ERROR BY MULTIPLICATION

N. C. Perry and J. C. Morelock, Alabama Polytechnic Institute

In applied mathematics the traditional approach to the analysis of the error in a result obtained by computation with approximate numbers is in terms of the maximum error possible [5]. The purpose of this paper is to illustrate a more statistical approach, which is presented briefly in Householder [2], and discussed more fully by Inman [3].* The general approach to be found in these references is the development of a distribution function which associates with an error of given size the probability of its being exceeded.

We illustrate this statistical point of view for the case of the multiplication of two approximate numbers. More precisely we have the following:

Problem: Given the numbers a and b, with respective errors ϵ and ϵ' which are on the range $-\frac{1}{2}$ to $+\frac{1}{2}$; to find the distribution function of the error E in ab, on the assumption that all values of ϵ and ϵ' are equally likely.

Since $(a+\epsilon)(b+\epsilon')=ab+a\epsilon'+b\epsilon+\epsilon\epsilon'$, the probability that E exceeds a fixed value K is equal to the $P[a\epsilon'+b\epsilon+\epsilon\epsilon'>K]$. If we choose an ϵ, then by solving the inequality $E>K$ it follows that $\epsilon'>(K-b\epsilon)/(a+\epsilon)$. Hence, the probability that simultaneously ϵ lies on a small interval of length $\Delta\epsilon$ and ϵ' lies on the interval $[(K-b\epsilon)/(a+\epsilon),\frac{1}{2}]$ is roughly equal to the product

$$(\Delta\epsilon)(\tfrac{1}{2}-(K-b\epsilon)/(a+\epsilon)).$$

With certain necessary modifications the essential plan of proof is to integrate this probability for ϵ ranging from $-\frac{1}{2}$ to $\frac{1}{2}$. Thus apparently, $P[E>K]$ is equal to the expression

* Specific applications to differential equations and matrix theory can be found in Rademacher [4], and Goldstine and von Neumann [1].

$$(1)\qquad \int_{-1/2}^{1/2}\left(\tfrac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon.$$

However, since ϵ' lies between $-\frac{1}{2}$ and $\frac{1}{2}$, the quantity $(K-b\epsilon)/(a+\epsilon)$ in the integrand of equation (1) must satisfy the inequality $-\frac{1}{2}\leqq(K-b\epsilon)/(a+\epsilon)\leqq\frac{1}{2}$. Upon solving for ϵ we obtain $(2K-a)/(2b+1)\leqq\epsilon\leqq(2K+a)/(2b-1)$. Thus, if $\epsilon>(2K+a)/(2b-1)$ we must replace $(K-b\epsilon)/(a+\epsilon)$ by $-\frac{1}{2}$, and similarly if $\epsilon<(2K-a)/(2b+1)$ we must replace $(K-b\epsilon)/(a+\epsilon)$ by $\frac{1}{2}$. More precisely $P[E>K]$ seems to take the form:

$$(2)\qquad \int_{-1/2}^{(2K-a)/(2b+1)}(\tfrac{1}{2}-\tfrac{1}{2})d\epsilon+\int_{(2K-a)/(2b+1)}^{(2K+a)/(2b-1)}\left(\tfrac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon+\int_{(2K+a)/(2b-1)}^{1/2}[\tfrac{1}{2}-(-\tfrac{1}{2})]d\epsilon.$$

However, if $K<(2a-2b-1)/4$ it is impossible for $(K-b\epsilon)/(a+\epsilon)$ to be greater than $\frac{1}{2}$, and the correction shown in (2) for the range $-\frac{1}{2}$ to $(2K-a)/(2b+1)$ becomes unnecessary. Similarly if $K>(2b-2a-1)/4$ then $(K-b\epsilon)/(a+\epsilon)$ cannot be less than $-\frac{1}{2}$, and the correction indicated in equation (2) for the range $(2K+a)/(2b-1)$ to $\frac{1}{2}$ is not needed. Thus we have finally the distribution function of E defined as follows:

$$\frac{-2a-2b+1}{4}\leqq K\leqq\frac{2a-2b-1}{4};$$

$$P[E>K]=\int_{-1/2}^{(2K+a)/(2b-1)}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon+\int_{(2K+a)/(2b-1)}^{1/2}d\epsilon$$

$$\frac{2a-2b-1}{4}\leqq K\leqq\frac{2b-2a-1}{4};$$

$$P[E>K]=\int_{(2K-a)/(2b+1)}^{(2K+a)/(2b-1)}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon+\int_{(2K+a)/(2b-1)}^{1/2}d\epsilon$$

$$\frac{2b-2a-1}{4}\leqq K\leqq\frac{2a+2b+1}{4};$$

$$P[E>K]=\int_{(2K-a)/(2b+1)}^{1/2}\left(\frac{1}{2}-\frac{K-b\epsilon}{a+\epsilon}\right)d\epsilon.$$

Upon performing these integrations we have:

$$P[E>K]=\frac{a+b}{2}+\frac{3}{4}+K-(K+ab)\log_e\frac{4(K+ab)}{(2a-1)(2b-1)};$$

$$\frac{-2a-2b+1}{4}\leqq K\leqq\frac{2a-2b-1}{4}$$

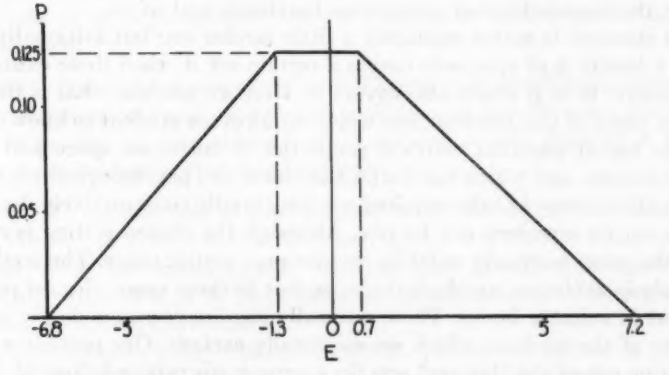
$$P[E > K] = a + \frac{1}{2} - (K + ab) \log_e \frac{2b+1}{2b-1};$$

$$\frac{2a-2b-1}{4} \leqq K \leqq \frac{2b-2a-1}{4}$$

$$P[E > K] = \frac{a+b}{2} + \frac{1}{4} - K - (K + ab) \log_e \frac{(2a+1)(2b+1)}{4(K+ab)};$$

$$\frac{2b-2a-1}{4} \leqq K \leqq \frac{2a+2b+1}{4}.$$

A general notion of the geometric nature of this function can be obtained from the graph below:



Graph of the frequency distribution of E for $a=6$, and $b=8$.

From these results it is readily computed that about half the time the error E in the product (6) (8) will lie outside the range $(-2.13, 2.09)$. Five percent of the time it will lie outside the range $(-5.32, 5.61)$, and about one percent of the time it will be less than -6.09 or greater than 6.50.

References

1. Goldstine, H. H. and von Neumann, J. Numerical inverting of matrices of high order, II, Proc. Am. Math. Soc., vol. 2, 1951, pp. 188–202.

2. Householder, A. S. Principles of Numerical Analysis, New York, 1953, Ch. 1.

3. Inman, S. The probability of a given error being exceeded in approximate computation, Math. Gaz., vol. 34, 1950, pp. 99–113.

4. Rademacher, H. A. On the accumulation of errors in processes of integration on high-speed calculating machines, Proc. Symposium Large Scale Digital Calculating Machinery, 176–185, Harvard University Press, Cambridge, Mass., 1948.

5. Scarborough, J. B. Numerical Mathematical Analysis, Baltimore, 1950, Ch. 1.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics. Massachusetts Institute of Technology, Cambridge 39, Mass.

THE HEINE-BOREL THEOREM

C. A. HAYES, JR., University of California at Davis

Mathematics is replete with "covering" theorems of one type or another, but the Heine-Borel theorem is perhaps the most well-known of them all. Virtually everyone who has taken a course in advanced calculus has encountered it, and has seen its applications in the proofs of theorems on uniform continuity, the boundedness of continuous functions, and so on.

The theorem is stated explicitly a little farther on; but informally, it says that if a family $\mathfrak{F}$ of open sets covers a certain set A, then there exists a finite subcollection $\mathfrak{G}$ of $\mathfrak{F}$ which also covers A. Here we propose what is thought to be a new proof of this theorem, one which requires the student to know or accept only the barest essential metrical properties of euclidean space and the real number system, and which has the philosophical and psychological advantage of proving the existence of the required covering family constructively, by actually picking out its members one by one. Although the chosen setting is euclidean space, the proof is equally valid in any compact metric space. The method does not apply in arbitrary topological spaces; but in these cases, elegant proofs can be found in suitable books. These naturally require correspondingly advanced maturity of the student, which we specifically exclude. Our procedure consists of selecting one of the "largest" sets (in a sense made precise below) of the given family; then from a certain subset of those remaining, one of the largest of that set is chosen, and so on. All that remains is to ascertain that the covering is completed after a finite number of such steps.

We introduce some definitions and notation. We let E_n denote n dimensional euclidean space; when points are mentioned, these will be understood to be points of E_n. If A and B are sets in E_n, then $A-B$ will denote the set of those points which are in A but not in B; the union of a number of sets will be denoted by $+$ or $\sum$ with appropriate indices; we write $A \subset B$ to denote the fact that A is contained in B. For any two points p and q, we let $\delta(p, q)$ denote the euclidean distance between them. For any given positive number η and any given point q, the set of those points p such that $\delta(q, p) < \eta$ is called the ***open sphere of center q and radius η***. A set G in E_n is called ***bounded*** if it is contained in some open sphere. A set G of points is called ***open*** if and only if every point q in G is the center of some open sphere $S(q, \eta)$ which is contained in G. It is easy to see that every open sphere is an open set. A point p is called a ***point of accumulation*** or a ***limit point*** of a set F in E_n if and only if every open sphere centered at p contains infinitely many points of F. The point p may or may not belong to F. In case

every limit point p of F does belong to F, we say that F is *closed*. Finally, if $\mathfrak{R}$ is a family of sets in E_n, and A is a set in E_n, we shall say that $\mathfrak{R}$ *covers* A if and only if A is contained in the union of $\mathfrak{R}$; that is, every point of A lies in at least one member of $\mathfrak{R}$.

A few properties of euclidean space and the real number system are needed in what follows. We assume known, or postulated, that every bounded infinite set of points in E_n has a limit point. We assume that the properties of least upper bounds are understood; in particular, that if γ is the least upper bound of a non-empty bounded set E of positive numbers, then there exists at least one number η in E such that $\eta > \gamma/2$. Finally, we use the fact that if $\alpha > 0$, and A is a set in E_n such that for each pair of different points p, q in A, $\delta(p, q) > \alpha$, then A has no limit point. This is easily seen, for if t is any point in E_n, then the distance between any two points of the sphere $S(t, \alpha/2)$ is less than α. Thus no such sphere can contain more than one point of A; A has no limit points.

We now prove a preliminary lemma which is, in fact, a special case of the main theorem.

LEMMA. *If A is a bounded closed set in E_n, and if $\mathfrak{R}_1$ is a family of open spheres of uniformly bounded radii, such that* (i) *each point in A is the center of at least one sphere in $\mathfrak{R}_1$, and* (ii) *each sphere in $\mathfrak{R}_1$ is centered at some point of A, then there exists a finite subfamily $\mathfrak{R}$ of $\mathfrak{R}_1$ which covers A.*

Proof. We assume that A has infinitely many points or there is nothing to prove. We denote by $r(S)$ the radius of any sphere S in $\mathfrak{R}_1$. If $\mathfrak{L}$ is any non-empty subfamily of $\mathfrak{R}_1$, we denote by $\rho(\mathfrak{L})$ the least upper bound of all numbers of the form $r(S)$, where S is in $\mathfrak{L}$. Since the radii of all spheres in $\mathfrak{R}_1$ are uniformly bounded, we have $0 < \rho(\mathfrak{L}) < \infty$.

We now construct the required family $\mathfrak{R}$. We select a sphere S_1 in $\mathfrak{R}_1$, with radius $r(S_1) > \rho(\mathfrak{R}_1)/2$ and center p_1 in A. If $A \subset S_1$, the lemma is proved. If not, then the set $A - S_1$ is non-empty, and we let $\mathfrak{R}_2$ denote the non-empty family of those spheres in $\mathfrak{R}_1$ whose centers are points of $A - S_1$. We select a sphere S_2 in $\mathfrak{R}_2$ with radius $r(S_2) > \rho(\mathfrak{R}_2)/2$ and center at p_2 in $A - S_1$. If $A \subset S_1 + S_2$, then the family consisting of S_1 and S_2 satisfies our requirements; otherwise we repeat the process on the non-empty family $\mathfrak{R}_3$ of those spheres in $\mathfrak{R}_1$ whose centers are points of $A - (S_1 + S_2)$, and continue thus inductively. In this way, we obtain a finite or infinite sequence of spheres $S_1, S_2, \cdots, S_m, \cdots$ with centers at $p_1, p_2, \cdots, p_m, \cdots$ respectively, drawn from the subfamilies $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_m, \cdots$ of $\mathfrak{R}_1$, in such a way that for each positive integer m:

(1) p_1 is in A; p_{m+1} is in $A - \sum_{i=1}^{m} S_i$;

(2) $0 < \rho(\mathfrak{R}_m)/2 < r(S_m) \leqq \rho(\mathfrak{R}_m)$.

By condition (1), for each pair of integers μ, ν, $\mu < \nu$ we have

(3) p_μ is the center of S_μ, p_ν is outside S_μ; $\delta(p_\mu, p_\nu) \geqq r(S_\mu)$.

We assert that the process outlined above must terminate for some integer N; that is, $A \subset \sum_{i=1}^{N} S_i$. Suppose the contrary, if possible. Let E denote the set of points $p_1, p_2, \cdots$. From (3), no two coincide, hence E is an infinite subset of

the bounded set A. Then E has at least one limit point p, which is *a fortiori* a limit point of the closed set A, so that p is also in A.

Either p belongs to some sphere S_m or not. If p were in S_μ, say, then since S_μ is open and p is a limit point of E, there must exist infinitely many points of E lying in S_μ; in particular there must be one such point p_ν with index $\nu > \mu$, contradicting (3). If p were not in any sphere S_m, then p must be itself the center of a sphere S belonging to each of the families $\mathfrak{R}_1, \mathfrak{R}_2, \cdots$. Hence, using (2) and (3), for $\mu, \nu = 1, 2, \cdots, \mu \neq \nu$, we should have

$$0 < r(S) \leqq \rho(\mathfrak{R}_\mu) < 2r(S_\mu); \qquad \delta(p_\mu, p_\nu) > r(S)/2 > 0.$$

This would mean E could have no limit points. This contradiction forces the conclusion $A \subset \sum_{i=1}^{N} S_i$ for some integer N; the family $\{S_1, S_2, \cdots, S_N\} = \mathfrak{R}$ covers A as required.

THEOREM. (Heine-Borel). *If $\mathfrak{G}_1$ is a family of open sets covering the bounded closed set A in E_n, then a finite subfamily $\mathfrak{G}$ of $\mathfrak{G}_1$ covers A.*

Proof. We let $\mathfrak{R}_1$ denote the family of open spheres of radii not exceeding 1, whose centers are points of A, and each of which is contained in some member of $\mathfrak{G}_1$. $\mathfrak{R}_1$ clearly satisfies the hypotheses of the preceding lemma. We thus find a finite family of spheres $S_1, S_2, \cdots, S_N$ in $\mathfrak{R}_1$ covering A. For each such sphere S_i we pick some set G_i in $\mathfrak{G}_i$ which contains S_i. The finite family $\{G_1, G_2, \cdots, G_N\} = \mathfrak{G}$ clearly covers A.

WHAT IS A POINT OF INFLECTION?

A. W. WALKER, University of Toronto

Has the curve $y = x^4$ a point of inflection at the origin? At present, the answer depends on whether we consider the matter from the point of view of calculus or algebraic geometry.

As Ewing [1] has remarked, considerable care is required in expressing the necessary and sufficient conditions for an inflection point in the language of the calculus, and inconsistencies are found when various texts are compared. However, the generally accepted view is surely the one expressed by von Mangoldt [2] in the section of the *Encyklopädie* on the application of calculus to curves and surfaces, namely, that a point of inflection is any point where a plane curve crosses both its tangent and its normal. (The curve may consist of two parts with different equations, joining, without discontinuity in the direction of the tangent line, at the point in question. An inflection point is not necessarily a point of zero curvature, and conversely.) According to this definition, the curve $y = x^4$ has, of course, no inflection points.

In algebraic geometry we are concerned to a considerable extent with complex or abstract fields, in which the above definition has no meaning.* The general viewpoint here (expressed with lack of clarity and consistency by many

* The writer is indebted to the referee for this significant remark.

texts in English) seems to be the one clearly stated by R. J. Walker [3]. After remarking that the terms *flex* and *point of inflection* are identical, he defines a flex of a plane curve as a non-singular point at which the tangent line has n coincident intersections with the curve, where $n \geqq 3$. Berzolari [4], in the section of the *Encyklopädie* on higher plane algebraic curves, refers to an *ordinary* inflection point when $n=3$, and to a *higher order* inflection point when $n>3$; in particular, the term *undulation point* is used when $n=4$. This is so for the curve $y=x^4$ at the origin. (It is instructive to consider $y=x^4-k^2x^2$ and let k approach zero.)

The only known reference to the fact that there are (in the real field) these two conflicting definitions is a short footnote in de la Vallée Poussin [5], which does not appear in the first edition but was added later. If the dual interpretation is to continue, it would avoid confusion if writers of future texts on both subjects followed this lead, preferably including a reference to [2] and [4] or their equivalents. An alternative would be the universal adoption in algebraic geometry of the concise term *flex*, defined as above. A flex would be either an inflection point (n odd) or an undulation point (n even), extending the scope of this latter term; the distinction would be significant only in the real field. (Salmon [6] remarks: "Cramer calls those points at which the tangent meets the curve in an odd number of consecutive points, points of visible inflection, to distinguish them from points of undulation, which do not, to the eye, differ from ordinary points on the curve.")

References

1. This MONTHLY, vol. 45, 1938, p. 681.
2. Encyk. der Math. Wiss., III, 3, p. 9.
3. R. J. Walker, Algebraic Curves, Princeton, 1950, p. 71.
4. Encyk. der Math. Wiss., III, 2(1), p. 320.
5. De la Vallée Poussin, Cours d'Analyse Infinitésimale, 7th ed., 1930, Vol. 1, p. 242.
6. Salmon, Higher Plane Curves, 3rd ed., 1879, p. 37.

HODOGRAPH OF A CENTRAL ORBIT

KULDIP SINGH, National Defence Academy, Dehra Dun, India

Let a particle move under a central force $-f(r)(\mathbf{r}/r)$ per unit mass where r is the distance of the particle from the center of force, $\mathbf{r}$ is the vector from the center of force to the particle, and $(\mathbf{r}/r)$ is the unit vector in this direction. Then the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -f(r)\frac{\mathbf{r}}{r}.$$

Now $r^2(d\theta/dt)=h$, therefore

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{r^2}{h}f(r)\frac{\mathbf{r}}{r}\frac{d\theta}{dt}.$$

Let $r=g(\theta)$ be the polar equation of the orbit with center of the force as the pole. If $\mathbf{R}$ is the position vector of any point on the hodograph (*i.e.*, $\mathbf{R}$ is the velocity vector), then

$$\frac{d\mathbf{R}}{dt} = -\frac{1}{h}F(\theta)(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\frac{d\theta}{dt},$$

where $F(\theta)=r^2f(r)$, $r=g(\theta)$. Hence,

$$\mathbf{R} = -\frac{1}{h}\int F(\theta)(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})d\theta + \mathbf{C},$$

which is the equation of the hodograph.

Example. A particle describes a conic under a central force to the focus; show that the hodograph is a circle.

Since

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^2}\frac{\mathbf{r}}{r},$$

$$\frac{d\mathbf{R}}{dt} = -\frac{\mu}{h}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\frac{d\theta}{dt}.$$

Therefore

$$\mathbf{R} = -\frac{\mu}{h}(\sin\theta\mathbf{i}-\cos\theta\mathbf{j}) + \mathbf{C}.$$

Hence the hodograph is the circle

$$\mathbf{R}^2 - 2\mathbf{R}\cdot\mathbf{C} + \left(\mathbf{C}^2 - \frac{\mu^2}{h^2}\right) = 0$$

whose center has the position vector $\mathbf{C}$ and whose radius is μ/h.

N. B. The procedure adopted in the above example can be usefully applied to most of the examples on hodograph.

PRINCIPAL VALUES OF CERTAIN INVERSE TRIGONOMETRIC FUNCTIONS

C. B. Read and Ferna Wrestler, University of Wichita

Students often feel the definitions of the principal values of the inverse trigonometric functions are purely arbitrary. Texts in trigonometry may fail to point out that alternative definitions exist; they sometimes justify the definitions given on the basis of later use in the calculus.

The calculus student in turn will be fortunate if his text explains why the particular definitions used were selected. He may for example wonder why $-\pi/2 \leqq \text{Arc sin } x \leqq \pi/2$, rather than $\pi/2 \leqq \text{Arc sin } x \leqq 3\pi/2$. The first choice results in a positive slope throughout the interval, in agreement with the usual

value of the derivative. This explanation may not be a satisfactory reason why, for $x \leqq -1$, we define $-\pi \leqq \text{Arc sec } x < -\pi/2$ rather than $\pi/2 < \text{Arc sec } x \leqq \pi$ since the second choice would give a positive slope throughout while the first does not. Some calculus texts give the second definition; to many students this seems more consistent since it agrees with the definition for Arc cos x.

Two suggestions are offered in justification of the choice, when $x \leqq -1$, $-\pi \leqq \text{Arc sec } x < -\pi/2$. If in the usual derivation of the derivative of Arc sec x we let $x = \sec y$, then $dx/dy = \sec y \tan y = x \tan y$. If $x \geqq 1$ and $0 \leqq y < \pi/2$, then $\tan y = \sqrt{x^2-1}$; if $x \leqq -1$ and $\pi/2 < y \leqq \pi$, then $\tan y$ is $-\sqrt{x^2-1}$, hence $dy/dx = \pm 1/x\sqrt{x^2-1}$, requiring the ambiguous sign. However, if when $x \leqq -1$ we define $-\pi \leqq y < -\pi/2$, $\tan y$ is $\sqrt{x^2-1}$ and for all cases $dy/dx = +1/x\sqrt{x^2-1}$ (dy/dx is not defined at $x = \pm 1$).

As the second suggestion, consider the area bounded by $y = 1/(x\sqrt{x^2-1})$ between $x = -2$ and $x = -\sqrt{2}$. Clearly the area is negative but if the principal value of Arc sec x is defined as $0 \leqq \text{Arc sec } x \leqq \pi$, a positive result is obtained, suggesting reasons for another definition.

These explanations may still leave the student (or the teacher) puzzled as to why the "natural" definition Arc sec $x = \text{Arc cos } (1/x)$ should not be adopted. The ambiguity of signs discussed in the first suggestion just presented can be readily removed by defining $0 \leqq \text{Arc sec } x = \text{Arc cos } (1/x) \leqq \pi$ and using absolute values in the formula

$$\frac{d}{dx} \text{Arc sec } x = \frac{1}{|x|\sqrt{x^2-1}}.$$

In the case of integration, we may write, for negative x,

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{d(-x)}{(-x)\sqrt{x^2-1}} = \text{Arc sec } (-x) + C \qquad x \leqq -1$$

$$= \int \frac{d|x|}{|x|\sqrt{x^2-1}} = \text{Arc sec } |x| + C$$

hence for x either positive or negative $\int dx/x\sqrt{x^2-1} = \text{Arc sec } |x| + C (|x| \geqq 1)$.

Use of this formula (analogous to $\int dx/x = \ln |x| + C$) and the definition $0 \leqq \text{Arc sec } x \leqq \pi$ gives, for the integration example presented, a negative result. It would appear, then, that there is really no necessity for an apparently artificial definition.

It would not be out of order to suggest that authors might well devote a little more space to explaining why the particular choice of definition was selected.

Note: The authors wish to thank the referee for a suggestion for expanding and improving this note.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1206. *Proposed by W. E. Briggs, University of Colorado*

If n is a given positive integer, how many integral solutions does

$$1/n = 1/x + 1/y$$

have with x and y positive and unequal?

E 1207. *Proposed by R. D. Gordon, California State Polytechnic College*

Given two mutually perpendicular lines l_1 and l_2 in a plane, and a point Q in the plane located equally distant from l_1 and l_2. Determine the locus of a point P in the plane if its distance from Q equals the sum of its distances from l_1 and l_2.

E 1208. *Proposed by Gretchen Geller, Student, University of California, Davis, Calif.*

Can a closed interval I be covered by an infinite set J of nondegenerate closed intervals in such a manner that no finite subset of J covers I?

E 1209. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

E 1210. *Proposed by Michael Goldberg, Washington, D. C.*

Given two equilateral triangles of edges a and b. Show how to dissect them by straight cuts into a total of no more than six pieces which can be assembled into another equilateral triangle. When the ratio of the larger to the smaller satisfies $a/b \geqq \sqrt{3}$, then five pieces suffice. When $a/b = 4/3$, four pieces suffice.

SOLUTIONS

Good, Better, Best

E 1176 [1955, 493]. *Proposed by J. L. Brenner, State College of Washington*

Given a, b, c, d real and $ad - bc = 1$. Show that

$$Q = a^2 + b^2 + c^2 + d^2 + ac + bd \neq 0, 1, -1.$$

I. *Solution by Calvin Foreman, Baker University.* We have

$$2Q - 2 = (a + c)^2 + (a - d)^2 + (b + d)^2 + (b + c)^2,$$

the right member of which cannot vanish, since then $c = -a = -d = b = -c$ requiring $a = b = c = d = 0$ which contradicts the stipulation that $ad - bc = 1$. Therefore $2Q - 2 > 0$, that is $Q > 1$, from which the desired conclusion follows.

II. *Solution by A. P. Boblett, U. S. Naval Ordnance Laboratory, Corona, Cal.* Since

$$Q - 3/2 = (a + c/2 - 3d/4)^2 + (b + 3c/4 + d/2)^2 + 3(c^2 + d^2)/16$$

it follows that $Q \geqq 3/2$.

III. *Solution by Leonard Carlitz, Duke University.* Actually $Q \geqq \sqrt{3}$, as is seen from the identity

$$Q - \sqrt{3} = (a + c/2 - d\sqrt{3}/2)^2 + (b + c\sqrt{3}/2 + d/2)^2.$$

That we can have $Q = \sqrt{3}$ is seen by taking $a^2 = \sqrt{3}/2$, $b = -a/\sqrt{3}$, $c = 0$, $d = 2a/\sqrt{3}$.

Also solved by G. E. Bardwell, Hüseyin Demir, I. A. Dodes, R. V. Esperti, Harley Flanders, H. M. Gehman, Michael Goldberg, A. J. Goldman, H. E. Gould, E. S. Grable, G. R. Grainger, D. S. Greenstein, W. G. Hanks, Vern Hoggatt, Raymond Huck, A. R. Hyde, L. E. Isenecker, J. D. E. Konhauser, A. E. Livingston, D. C. B. Marsh, W. H. McKenzie, J. D. Miller, F. D. Parker, Kincaid Patterson, Larry Potter and J. T. Robertson (jointly), C. E. Rhodes, D. W. Smith, J. A. Tierney, Gene Uretz, Chih-yi Wang, and Charlotte Yesselman. Late solutions by Servet Aybat, Walter Guber and Sheldon Weinberg (jointly), M. J. Hellman, and M. A. Rashid.

Three Associated Loci

E 1177 [1955, 493]. *Proposed by W. R. Utz, University of Missouri*

Describe the three types of plane loci of points the product of whose distances from a point and a line is constant.

Solution by D. C. B. Marsh, Colorado School of Mines. Let the fixed line be the y-axis and the fixed point the point $(2a, 0)$ and denote the constant product by k. Then

$$x^2y^2 + x^2(x - 2a)^2 = k^2$$

is the equation of the desired locus.

The locus is symmetric in the x-axis, with y-axis as vertical asymptote and real intercepts on the x-axis for abscissae $a + (a^2 \pm k)^{1/2}$ and/or $a - (a^2 \pm k)^{1/2}$ which are real; excluded intervals likewise depend on the relative sizes of k and a^2. (1) If $k < a^2$, the curve consists of two branches asymptotic to the y-axis

and an isolated oval surrounding the fixed point; (2) if $k=a^2$, the curve consists of two branches asymptotic to the y-axis, one of which has a loop on the x-axis; in the third case, (3), where $k>a^2$, the loop has reduced to a point and the curve consists of two asymptotic branches only. A noteworthy similarity exists between these curves and the conchoids of Nicomedes.

Also solved by Michael Goldberg, A. R. Hyde, Larry Wan, and the proposer.

The Proposer has found this problem an interesting one to pose to students in or beyond analytic geometry as it replaces *quotient* in the definition of the conics by *product*. Hyde presented carefully plotted graphs of the three types of loci and further distinguished a critical case for type (3) above. When $a^2<k<3\sqrt{3}a^2/4$ the curve possesses four real horizontal tangents, when $k=3\sqrt{3}a^2/4$ there are two real horizontal (inflectional) tangents, and when $k>3\sqrt{3}a^2/4$ there are no real horizontal tangents.

An Arithmetical Property of the Triangle

E 1178 [1955, 493]. *Proposed by A. J. Goldman, Princeton University*

Prove that there exists a positive constant c with the following property: If T is any triangle whose area exceeds c, then the product of the lengths of the sides of T is greater than the area of T. What is the best possible value of c?

Solution by Calvin Foreman, Baker University. The product P of the lengths of the sides of T is greater than the area K of T if, and only if, the circumradius R exceeds $\frac{1}{4}$ since $P=4KR$. The circumradius is minimal and equals $\frac{1}{4}$ for a triangle of prescribed area when the triangle is an equilateral triangle whose area equals $(3/64)\sqrt{3}$. The desired conclusion follows, and the best value of c is $(3/64)\sqrt{3}$.

Also solved by Fred Cherry, Michael Goldberg, D. S. Greenstein, R. E. Heaton, Vern Hoggatt, A. R. Hyde, Sam Kravitz, C. W. Langley, D. C. B. Marsh, C. M. Sandwick, Sr., Chi-yi Wang, and the proposer. Late solutions by C. H. Borgmeyer, John Christopher, C. F. Pinzka, and J. W. Ross.

Limit of an Iterated Operation

E 1179 [1955, 493]. *Proposed by C. D. Olds, San Jose State College*

Let f be an operator such that $f(z)=(|z|+z)/2$, and define $f^2(z)=f\{f(z)\}$, $\cdots$, $f^n(z)=f\{f^{n-1}(z)\}$. Evaluate

$$\lim_{n\to\infty} f^n(i), \text{ where } i=\sqrt{-1}.$$

I. *Solution by Michael Goldberg, Washington, D. C.* The procedure can be visualized by the use of the Argand diagram. The value f^1 is midway between 1 and i; f^2 is midway between f^1 and $|f^1|$; *etc.* Therefore

$$|f^n| = |f^{n-1}| \cos(\pi/2^{n+1}).$$

The limiting value is real and is evaluated by the well known infinite product

$$\cos \pi/4 \cos \pi/8 \cos \pi/16 \cdots = 2/\pi.$$

II. *Solution by Michel McKiernan and Berthold Schweizer, Illinois Institute of Technology.* The problem may be solved for any z. Let $z = r \exp(i\theta)$. By constructing the vector $f(z)$ in the complex plane it is readily seen that

$$f(z) = r \cos(\theta/2) \exp(i\theta/2),$$

and that, in general,

$$f^n(z) = r\left[\prod_{k=1}^{n} \cos(\theta/2^k)\right] \exp(i\,\theta/2^n).$$

Since (see, *e.g.*, Knopp, *Infinite Series*, p. 228)

$$\prod_{k=1}^{\infty} \cos(\theta/2^k) = (\sin\theta)/\theta$$

we have

$$\lim_{n\to\infty} f^n(z) = (r \sin\theta)/\theta,$$

and in particular,

$$\lim_{n\to\infty} f^n(i) = 2/\pi.$$

Also solved by W. A. Al-Salam, G. B. Charlesworth, Marius Cohn, Hüseyin Demir, A. L. Epstein, Thomas Erber, R. V. Esperti, Calvin Foreman, A. J. Goldman, D. S. Greenstein, J. R. Hatcher, R. E. Heaton, A. R. Hyde, Rufus Isaacs, P. G. Kirmser, J. D. E. Konhauser, D. C. B. Marsh, W. H. McKenzie, C. S. Ogilvy, F. D. Parker, Paul Payette, C. A. Rogers, B. D. Seckler, Eugene Usdin, Chih-yi Wang, J. V. Whittaker, and the proposer. Late solutions by T. Y. Chow, P. B. Johnson, and M. S. Klamkin.

Editorial Note. The limit of the infinite product is apparent from

$$\frac{\sin\theta}{\theta} = \cos(\theta/2)\frac{\sin(\theta/2)}{\theta/2} = \cos(\theta/2)\cos(\theta/4)\frac{\sin(\theta/4)}{\theta/4}$$

$$= \cdots = \cos(\theta/2)\cos(\theta/4)\cdots\cos(\theta/2^k)\frac{\sin(\theta/2^k)}{\theta/2^k}.$$

The problem is intimately tied up with Vieta's famous infinite product for $2/\pi$ and also with the well known quadratrix of Hippias and Dinostratus.

The 2319th Digit in 1000!

E 1180 [1955, 493]. *Proposed by M. S. Klamkin and Alex Kraus, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

Solution by J. B. Muskat, Cambridge, Mass. By Stirling's formula we find that the expansion of 1000! has 2568 digits. Since 5 divides 1000! 249 times, and 2 divides 1000! more than 500 times, the expansion of 1000! terminates in a string of 249 zeros. Further, since $2568-249=2319$, the digit we seek is the last non-zero digit in the expansion. Clearly this digit is even, and depends upon the last digits of the factors 1, 2, 3, $\cdots$, 1000 of 1000! after removal of all factors 5. Taking the factors successively in sets of ten and removing the multiples of 5 we have, for the product of the final digits of the remaining factors in each set,

$$1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdot 9 \equiv 6 \pmod{10}. \tag{1}$$

There are 100 of these products, leaving 200 numbers which are divisible by 5, and if 5 is factored out of each, 200! remains. We then have 20 more products like (1), leaving 40 numbers which are divisible by 5, and if 5 is factored out of each of these, 40! remains. We get 4 more products like (1), leaving 8 numbers which are divisible by 5, and if 5 is factored out of each of these, 8! remains. Dividing out 5 once more, we have left

$$1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8 \equiv 4 \pmod{10}.$$

Now $4\cdot 6^{124}\equiv 4 \pmod{10}$, whence 4 is the last digit in $(1000!)/5^{249}$. But 2^{249} must also be divided out to complement the 5^{249}. Since $2^{249}\equiv 2 \pmod{10}$, it now follows that the last digit in $(1000!)/10^{249}$, which is the digit we are seeking, is 2.

Also solved similarly by the proposers. Different (in some cases incorrect or incomplete) solutions were given by G. E. Bardwell, Hüseyin Demir, G. B. Charlesworth, Fred Cherry, Michael Goldberg, A. J. Goldman, A. R. Hyde, C. W. Langley, D. C. B. Marsh, C. S. Ogilvy, Paul Payette, Walter Penney, B. E. Rhoades, C. M. Sandwick, Sr., and Chih-yi Wang. Late communications from H. F. Bennett, John Christopher, Jerome Manheim, and Harold Saks.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4670 [1956, 47]. *Correction.* For $(0, \infty)$ read $[0, \infty)$.

4671 [1956, 47]. *Correction.*

The factor a^{2n+1} should appear in the right member of the required relation.

4678. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Evaluate

$$\tan^{-1} 1 - \tan^{-1} \tfrac{1}{3} + \tan^{-1} \tfrac{1}{5} - \cdots .$$

4679. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

If $A_1A_2A_3A_4A_5$ is a cyclic pentagon and if Ω_{ij} denotes the orthopole of the line A_iA_j with respect to the triangle formed by the remaining three vertices, then prove that the ten points Ω_{ij} all lie on a circle.

4680. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Solve the following generalization of Clairaut's differential equation

$$y - xy' + \frac{x^2y''}{2!} + \cdots (-1)^{n-1} \frac{x^{n-1}y^{(n-1)}}{(n-1)!} + (-1)^n \frac{x^nF(y^{(n)})}{n!} = G(y^{(n)}).$$

4681. *Proposed by Jack Klugerman, Evans Signal Laboratory, Belmar, N. J.*

Given a real symmetric matrix A which is triple diagonal, *i.e.*, it has a diagonal, an upper diagonal, a lower diagonal, and the remaining elements are zero; if c is the eigenvalue with highest multiplicity m, then there must be at least $m-1$ zeros in the upper diagonal.

4682. *Proposed by R. C. Lyness, Preston, England*

(a) Prove that when the series

$$1 + \sum_{r=1}^{\infty} \binom{r\alpha}{r-1} \frac{x^r}{r}$$

is convergent, its sum, y, satisfies $y = 1 + xy^\alpha$.

(b) Prove also that

$$\sum_{r=1}^{\infty} \binom{r\alpha + \beta - 1}{r-1} \frac{x^r}{r} = \frac{y^\beta - 1}{\beta}.$$

SOLUTIONS

An Invalid Inequality

4603 [1954, 571]. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i = 1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geqq \frac{n}{2},$$

equality occurring only if all denominators are equal.

Note by F. H. Northover, Memorial University of Newfoundland. I am indebted to Professor M. J. Lighthill, F.R.S., of Manchester University for the following counter-example.

Let $n=20$ and take $x_1, x_2, \cdots, x_{20}$ (in that order) to be $1+5\epsilon$, 6ϵ, $1+4\epsilon$, 5ϵ, $1+3\epsilon$, 4ϵ, $1+2\epsilon$, 3ϵ, $1+\epsilon$, 2ϵ, $1+2\epsilon$, ϵ, $1+3\epsilon$, 2ϵ, $1+4\epsilon$, 3ϵ, $1+5\epsilon$, 4ϵ, $1+6\epsilon$, 5ϵ; where ϵ is small. We can easily establish

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{19}}{x_{20}+x_1}+\frac{x_{20}}{x_1+x_2}=10-\epsilon^2+O(\epsilon^3),$$

and so, for ϵ small enough, the expression is less than $n/2=10$.*

Simultaneously Vanishing Real Functions

4617 [1954, 719; 1955, 186]. *Proposed by Albert Wilansky, Lehigh University*

Let $\{u_n\}$, $\{v_n\}$ be real sequences, $n=2, 3, 4, \cdots$, such that

$$\sum (u_n^2+v_n^2)<1.$$

Let f, g be real continuous functions of two real variables x, y, $x^2+y^2\leqq 1$, such that

$$f(\cos\theta, \sin\theta)=\cos\theta-\sum (u_n\cos n\theta+v_n\sin n\theta)$$

$$g(\cos\theta, \sin\theta)=\sin\theta-\sum (v_n\cos n\theta-u_n\sin n\theta).$$

Show that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2<1$.

Solution by the Proposer. The hypothesis on u_n, v_n should read $\sum(u_n^2+v_n^2)^{1/2}<1$. With this change, let

$$h(z)=\begin{cases} z-\sum_{n=2}^{\infty}(u_n+iv_n)z^{-n} & \text{for } |z|\geqq 1 \\ f(x, y)+ig(x, y) & \text{for } |z|=|x+iy|\leqq 1. \end{cases}$$

Then h is continuous on the complex plane, $h(z)/z\to 1$ as $z\to\infty$, and $h(z)$ has no zeros for $|z|\geqq 1$ since the hypotheses ensure that, for such z, $|h(z)|>|z|-1\geqq 0$. By a well-known result (*e.g.*, problem 4475 [1953, 271]), h has a zero. This gives the result. The method can obviously be applied to more complicated examples.

Also solved by A. J. Goldman and J. J. Kohn who remark that, with the original hypotheses it can only be shown that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2\leq 1$.

---

* The proposer submitted proof that the inequality is true for $n=3$, 4, but confessed that he had not found a proof for greater values. C. R. Phelps proved the case $n=5$. Nine alleged "proofs" for general n were received by the editor.

Hermitian Matrices

4618 [1955, 45]. *Proposed by A. Oppenheim, University of Malaya, Singapore*

$A=(a_{ik})$, $B=(b_{ik})$ are two non-negative definite Hermitian matrices of order n. C_n is a matrix whose elements are $a_{ik}^n+b_{ik}^n$. If a, b, c_n are the determinants of A, B, C_n, prove that $c_n \geqq (a+b)^n$.

Solution by the Proposer. Let $E=(e_{ik})$, $F=(f_{ik})$ be non-negative definite Hermitian matrices. Then $G=(e_{ik}\cdot f_{ik})$ and $H=(e_{ik}+f_{ik})$ are both non-negative definite Hermitian matrices and their determinants satisfy the inequalities

$$g \geqq ef, \tag{1}$$

$$h^{1/n} \geqq e^{1/n} + f^{1/n}. \tag{2}$$

(See *Journal London Math. Soc.* 5(1930), 114–119 for references and proofs.) Let $A=(a_{ik})$, $B=(b_{ik})$ and write

$$A_m = (a_{ik}^m), \qquad B_m = (b_{ik}^m), \qquad C_m = A_m + B_m, \qquad (m = 1, 2, \cdots).$$

If the respective determinants are a_m, b_m, c_m, then, by repeated application of (1),

$$a_m \geqq a^m, \qquad b_m \geqq b^m,$$

and, by (2),

$$c_m^{1/n} \geqq a_m^{1/n} + b_m^{1/n} \geqq a^{m/n} + b^{m/n}.$$

In particular, with $m=n$, $c_n \geqq (a+b)^n$, the inequality of the problem.

A Generalization of Ptolomey's Theorem

4619 [1955, 45]. *Proposed by V. F. Ivanoff, San Francisco, California*

Let L_i, $i=1, 2, \cdots, 5$, be five lines in space. Denote the angle between L_i and L_j by (ij). Prove that the determinant

$$\left|\sin^2 \frac{(ij)}{2}\right| = 0 \qquad (i, j = 1, 2, \cdots, 5).$$

Solution by Louise A. Wolf, University of Wisconsin. We have

$$\sin^2 \frac{(ij)}{2} = \frac{1-\cos(ij)}{2} = \tfrac{1}{2}(1 - x_ix_j - y_iy_j - z_iz_j),$$

where x_i, y_i, z_i are the direction cosines of L_i. Thus

$$|A| = \left|\sin^2 \frac{(ij)}{2}\right| = \frac{1}{32}\left|1 - x_ix_j - y_iy_j - z_iz_j\right|,$$

which can be expressed as the product of the two singular determinants

$$|A| = \frac{1}{32}\begin{vmatrix} 1 & x_1 & y_1 & z_1 & 0 \\ 1 & x_2 & y_2 & z_2 & 0 \\ 1 & x_3 & y_3 & z_3 & 0 \\ 1 & x_4 & y_4 & z_4 & 0 \\ 1 & x_5 & y_5 & z_5 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -x_1 & -x_2 & -x_3 & -x_4 & -x_5 \\ -y_1 & -y_2 & -y_3 & -y_4 & -y_5 \\ -z_1 & -z_2 & -z_3 & -z_4 & -z_5 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

and hence $|A| = 0$.

Also solved by Robert Breusch, William Kruskal, Bela Sz.-Nagy, H. A. Osborn, M. F. Smiley, J. A. Tierney, Chih-yi Wang, and the Proposer.

Editorial Note. The Proposer notes that the corresponding determinant of the fourth order vanishes if $L_1, \cdots, L_4$ lie in a plane. In fact the problem can be traced to the following problem in Salmon, *Conic Sections* (1879), p. 134: To find the relation connecting the mutual distances of four points on a circle. The answer is $|(ij)^2| = 0$, where (ij) is the distance between points P_i and P_j. What about more than three dimensions?

An Infinite Radical

4620 [1955, 45]. *Proposed by Herman Hanisch, New York University*

Evaluate

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}.$$

Solution by N. J. Fine, University of Pennsylvania. For $x>0$, let

$$f_n(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+\cdots+(x+n-1)\sqrt{1+(x+n)}}}}.$$

Clearly $f_n(x)$ increases with n. But

$$\begin{aligned} x+1 &= \sqrt{1+x(x+2)} \\ &= \sqrt{1+x\sqrt{1+(x+1)(x+3)}} \\ &= \cdots \\ &= \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+\cdots+(x+n-1)(x+n+1)}}} > f_n(x). \end{aligned}$$

Hence $f(x) = \lim_{n\to\infty} f_n(x)$ exists, and $f(x) \leqq x+1$. Also

$$f(x) > \sqrt{1+x\sqrt{1+x\sqrt{1+x\sqrt{\cdots}}}} = \frac{x+\sqrt{x^2+4}}{2} > x.$$

Thus $x < f(x) \leqq x+1$. Let $\delta(x) = x+1-f(x)$. Then $0 \leqq \delta(x) < 1$. From the easily derived functional equation

$$f^2(x) = 1 + xf(x+1)$$

we get

$$(x + 1 + f(x))\delta(x) = x\delta(x+1).$$

Hence

$$0 \leqq \frac{\delta(x)}{x} \leqq \frac{\delta(x+1)}{x+1} \leqq \cdots \leqq \frac{\delta(x+n)}{x+n} < \frac{1}{x+n} \to 0.$$

So $f(x) = x+1$ and the given expression $= f(2) = 3$.

Also solved by Robert Breusch, W. B. Carver, M. S. Cohen and Cornelius Groenewoud, Gerald Freilich, Emil Grosswald, H. M. Gurk, Edgar Karst, M. A. Kirchberg, Eugene Levin, D. J. Newman, F. H. Northover, F. R. Olson, Walter Penney, C. R. Phelps, W. G. Preble, H. D. Ruderman, Daniel Shanks, M. R. Spiegel, O. E. Stanaitis, T. Vij, Vladeta Vučković, J. V. Whittaker, and Koichi Yamamoto.

Editorial Note. Yamamoto mentions that the problem was proposed by S. Ramanujan. See *Collected Papers*, Cambridge 1927, p. 323, Question 289 (*Journ. Ind. Math. Soc.*, III, 90). For this and similar problems, refer to a paper, *On infinite radicals*, by Aaron Herschfeld, this MONTHLY, August–September 1935, pp. 419–429.

A Diophantine Problem

4621 [1955, 45]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For each of the following equations

$$x(3x + 1)(6x + 1) = ky^2, \qquad k = 1, 2, \cdots, 6,$$

find the solutions in positive integers or prove it insoluble.

Solution by W. J. Blundon, Memorial University of Newfoundland. Since x, $3x+1$, $6x+1$ are relatively prime in pairs, their square-free factors are given by the following table:

Case	k	x	$3x+1$	$6x+1$	*Case*	k	x	$3x+1$	$6x+1$
I	1	1	1	1	VI	5	5	1	1
II	2	2	1	1	VII	5	1	5	1
III	2	1	2	1	VIII	5	1	1	5
IV	3	3	1	1	IX	6	3	2	1
V	4	1	1	1	X	6	6	1	1

If $6x+1 = 5m^2$, or $3x+1 = 5m^2$, or $3x+1 = 2m^2$, then $2m^2 \equiv 1 \pmod 3$ which is clearly impossible. Therefore cases III, VII, VIII, IX all fail, and it follows that

$3x+1$ and $6x+1$ are perfect squares. Let $6x+1=X^2$. Then the original equation becomes

$$X^6 - X^2 = 12ky^2 \qquad k = 1, 2, 3, 4, 5, 6.$$

Let a be the product of the square-free factors of $12k$. Then the equation reduces to the form $x^4-1=aZ^2$, where a is given by

Case	I	II	IV	V	VI	X
a	3	6	1	3	15	2

Now it is known that $X^4-Y^4=aZ^2$ has no integral solutions (other than $X=Y$, $Z=0$) when $a=1, 2, 3$. Therefore cases I, IV, X fail. The only integral solution for $X^4-Y^4=15Z^2$ with $Y=1$ is (2, 1, 1) which is clearly inadmissible here. In the remaining case II, $X^4-Y^4=6Z^2$ has only one solution when $Y=1$, namely (7, 1, 20). Thus the given equations have no solutions unless $k=2$, and for this value, the unique solution $x=8$.

Also solved (partially) by D. C. B. Marsh and R. Venkatachalam Iyer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Universal Mathematics. Part I. Functions and Limits.* Preliminary Edition. By the 1954 Summer Writing Group of the University of Kansas. Lawrence, Kansas, Student Union Book Store. 1954. 10+310 pages. Planographed. $2.75.

This book is based upon an outline prepared by a committee of the Mathematical Association of America which was appointed to study the undergraduate program in mathematics and to make recommendations relating thereto. *Part I, Functions and Limits*, encompasses only the first semester's work of the freshman year and is intended for a course meeting three times per week. It is to be followed by *Part II, Structures in Sets*, which is to constitute the second semester's work. As the name implies, *Universal Mathematics* is intended to be used in a basic course in mathematics for all students, including engineering students. It is designed for students with normal high school preparation consisting of at least two years of high school mathematics, including intermediate algebra.

The book deviates sharply from other books on first year college mathematics both in its organization and in its content. In organizing their material the

* *Editorial Note:* In connection with this review the reader should refer to the following account by Professor Duren of his experiences in teaching this book at Tulane.

authors have given parallel presentations of almost every topic covered, one treatment being essentially intuitive, and the other a formal and more difficult treatment. It is not intended that the formal treatment be followed continuously but that appropriate selections of theorems and proofs be made by the instructor.

More emphasis is placed upon concepts and ideas than is customary in books at this level and the authors recognize that the average student with only normal preparation will not acquire from this book alone the manipulative techniques which are necessary for the further study of calculus and analytic geometry. The authors suggest that this situation be met by adding a "Technical Laboratory" course to the curriculum of the freshman year. This course would meet two hours per week throughout the year and would be intended for engineering students and other students who plan to take calculus and analytic geometry during their sophomore year.

The many departures from tradition in this book are not suggested by the chapter headings, which are as follows: I Coordinate Systems. II Scientific Measurements. III Functions. IV Limits. V Derivatives. VI Integrals. VII The Logarithmic and Exponential Functions. VIII Summary. In addition there is a fifteen page introduction and an eight page appendix. The Introduction contains an interesting discussion of the relationship of mathematical theories to the natural universe. The difference between empirical and mathematical proofs, the role of intuitive reasoning in mathematics, and the need for further mathematical theories are also discussed in the Introduction. Some of the ideas in the Introduction are followed up in the second chapter, where it is explained how real numbers and vectors are used for the measurement of physical quantities.

Throughout the book a serious effort has been made to present real mathematics at an elementary level. This is exemplified in the first chapter, where a careful discussion of the real number line by means of postulates is given. The real numbers are represented as infinite decimals and the postulate for a least upper bound is invoked to insure their existence. Further properties of real numbers, based upon the postulates for a complete ordered field, are given in the appendix. The treatment of real numbers paves the way for a discussion of approximate measurements and intervals, and for a discussion of the plane and its Cartesian coordinates. Vectors are introduced by means of parallel displacements and play a prominent role in the mathematical description of the plane. The usual analytic Euclidean geometry of the plane is by-passed in favor of a geometry of graphs (in which distances not measured along the axes, angles, circles, and arc length are undefined concepts). A relation is defined as a subset of the pairs which form the Cartesian product of two sets, and graphs of various kinds of relations are constructed.

Functions are introduced as special (single-valued) relations. A careful distinction is made between a function f and its functional value $f[a]$ at a. Brackets are used to indicate functional values whereas parentheses designate composite

functions. The letter x, when placed in parentheses after another function, is used to represent the identity function on a set X, so that if a is an element of X we have $f(x)[a]=f[x[a]]=f[a]$, and there is no logical distinction between f and $f(x)$. Polynomial functions and the algebra of these functions are discussed with reasonable thoroughness. A minor criticism might be made that it is neither proved nor stated that a polynomial function is expressible uniquely, and that without such uniqueness the concept of the degree of a polynomial is not well-defined. Convex sets on a line and in the plane, as well as convex functions, are also discussed in the chapter on functions.

The theory of limits which is used is based upon the Moore-Smith definition of a limit, which depends upon the properties of partially ordered sets. One might believe that these concepts would prove to be bitter pills for most beginning students. However this may be, and the reviewer is not too sanguine, the pills are sugar-coated with good illustrative examples and meticulous explanations. The more usual epsilon definition of a limit is given in the guise of a theorem. Limits of functions and continuous functions are defined and discussed for functions whose domains are convex sets. The theory of limits which is adopted leads to an elegant theory of the derivative and the definite integral.

The divided difference of a function over a neighborhood in its domain, which in a sense is a generalization of the familiar difference quotient, is fundamental to the authors' definition of a derivative. This divided difference is in fact an interval in which each point designates a difference quotient. The derivative is then defined as the final residue which is contained in all of the divided differences which are generated by a certain convergent limit process. If the limit process does not converge the final residue is a proper interval rather than a point, in which case the derivative does not exist. Even in this case the divided difference is a useful concept for the study of functions. These ideas are of course expressed in a precise mathematical form in the book. Standard topics such as the mean value theorem, maxima and minima, rates, motion, and inverse functions are in no way neglected.

The chapter on integration begins with a discussion of partitions, step functions, areas bounded by step functions, and Riemann sums. The Riemann sum which is defined, and which is fundamental for the definition of the definite integral, is not the classical Riemann sum, but is defined as a certain interval in which each point designates a classical Riemann sum. This makes possible the definition of a definite integral as a Moore-Smith limit of Riemann sums. Proofs of the fundamental theorem of the integral calculus, the substitution theorem, and the theorem for the existence of a definite integral of a continuous function are then given with an economy of effort. Existence theorems for definite integrals of monotone functions and sectionally integrable functions are seldom or never discussed in elementary courses, but they are proved in this book. The logarithmic function is defined by means of a definite integral and e is defined by the equation $\ln[e]=1$. It is shown that e^x is the inverse function of $\ln(x)$ and the properties of $\ln(x)$ and e^x are derived.

The authors have indicated on the cover of the book that it is "a book of experimental text materials." Trial and experiment will determine how much of the material can be effectively incorporated into courses designed for a large number of beginning students. The formal exposition, which is in a fairly concise mathematical style, will probably require, for the most part, more intellectual effort and innate mathematical ability than can reasonably be expected of most beginning students. But under the guidance of a experienced teacher the more able and ambitious students can study much of this material profitably. A larger number will be able to profit from the more discursive intuitive treatment upon which most of the class work is supposed to be based. This treatment makes use of diagrams, figures, and good illustrative examples wherever these aids are helpful. Both the formal and intuitive treatments include adequate lists of well chosen exercises. In the opinion of the reviewer the book would be easier to read if the formal parts were placed in appendices at the ends of the various chapters instead of being interspersed with the intuitive parts.

In spite of the general excellence of the exposition the reviewer believes that even the intuitive treatment will be difficult for most students to follow and that the instructor will not always find it easy to present the material effectively. But, as Professor Alfred Whitehead once wrote: "Whenever a textbook is written of real educational worth, you may be quite certain that some reviewer will say that it will be too difficult to teach from it. Of course it will be difficult to teach from it. If it were easy the book ought to be burned; for it cannot be educational."

Whether or not he expects to experiment with a course organized along the lines of this book the teacher of mathematics will do well to examine the book carefully. Its treatment of classical subject matter from a fresh and modern viewpoint will give him a deeper insight into his subject which cannot fail to have an impact upon his teaching.

H. P. Evans
University of Wisconsin

TULANE EXPERIENCE WITH UNIVERSAL MATHEMATICS, PART I

The University of Kansas and the Social Science Research Council supported in 1954 a Kansas Summer Writing Group which wrote a book of experimental text materials called: *Universal Mathematics, Part I*, which was an effort to put into concrete form some of the results of the studies of the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America. This hastily written preliminary edition was given a mass trial at Tulane University as a universal mathematics course for all first year students who do not have to take intermediate algebra. Some 750 students in engineering, liberal arts, and business administration were involved. About 28 instructors, ranging from graduate teaching assistants to full professors, took part in the trial. Recently, a staff meeting was held to discuss the results.

The book is not yet suitable as a textbook and caused considerable difficulty to students and instructors. The main trouble is that students cannot read it. However, the chairman's proposal to replace it with one of the existing mathematical analysis texts met with unanimous opposition. Most of the staff wanted radical revisions of the book. Moreover, the failure rates using this text were higher than normal, although the conclusion that this was due entirely to the text is unwarranted since the quality of freshman classes was poorer than usual.

One thing is clear; and that is the fact that *Universal Mathematics* is not adaptable to students whose high school background in mathematics is scant. It was not intended to be "universal" in the sense that everybody should take this course at the beginning of the first college year, but universal in the sense that is for everybody who continues mathematics after acquiring competence in intermediate algebra and geometry. Thus some students are ready for *Universal Mathematics* while still in school and others are not ready for it when they enter college. It was written to what the Committee judged to be realistic prerequisites for a beginning first year college student whose high school training properly qualifies him for the traditional first college courses in mathematics.

In judging the value of *Universal Mathematics, Part I*, it should be emphasized that it was not intended to be a textbook in the proper sense. For one thing, it includes entirely too much material, too many different ideas, for a textbook. A proper textbook should be much simpler. This book was intended as a kind of mother of textbooks. Another aspect of it, which cannot be overlooked, is the unusual method by which it was uncompromisingly written. The book was written with two relatively complete and independent presentations. First a formal account was written to establish the mathematical structure of the work. After the formal account, with its postulates, definitions, theorems, and proofs was complete, an intuitive presentation was written to follow it closely in the same order. The two accounts appear side by side in the book. The intuitive account is used in the classroom, with the formal one as a reference. This method has several advantages and disadvantages. With the complete mathematical theory established and made dominant by principle, the textbook writer's usual freedom to choose his material and to choose the order of presentation of his topics is severely limited. It requires much more imagination to find appropriate motivations and interpretations for young students if mathematical theory determines the formulations of concepts and their order of introduction, while serious teaching purpose dictates that the concepts shall be well motivated and supported by meaningful exercises so as to become part of the student's working mental equipment. However, it turns out that if the really general and abstractly simple form of the theoretical ideas is chosen then the interpretation becomes surprisingly easy and rich.

Thus the advantages of the dual presentation, mathematical and intuitive, lie in the fact that every intuitive or incomplete exposition will definitely permit

expansion and elaboration in a correct formal theory which has been set down in axioms. Moreover the "hard" theory is out of the way so that the "easier" intuitive account can be read by itself, with the student and teacher, instead of the textbook writer, determining when they shall turn to the more austere formal theory; yet the formal theory fits with the intuitive account both in notation and structure so that this desirable transition is facilitated. The Tulane experience indicates that this was realized in practice. A disadvantage of the method is that the first stage of intuitive simplification here presented is still too formal. It needs at least two more stages of simplification and many more exercises. Another aspect of the same difficulty is concerned with the compromise between a correct formal mathematical language and the common classroom jargon of elementary mathematics. This method of writing does not facilitate the making of such a compromise and results in difficult reading even on the intuitive side.

The excess of ideas had interesting effects. The early introduction of relations and vector concepts which was a kind of side issue seemed to appeal to many instructors and students, though others did not like it. In general, one class would take hold of one set of ideas, and another class a different one. Some instructors found the Introduction very stimulating and used it as a part of the text material. The radical method of presenting limits was well liked by some and disliked by other instructors. One set of ideas seems to have been almost universally disliked, and that was the treatment of the divided difference as an interval of numbers. The chapter on scientific measurement was omitted to shorten the course. The fact that the book presents more ideas than is considered desirable caused unfavorable first reactions on teachers. Mathematics textbooks are required by convention to be as empty as possible of ideas, certainly empty in comparison with a textbook in chemistry, such as that of Pauling, or a first college textbook in physics. The Tulane trial seems to indicate that, although an idea-packed book requires that the teacher help the students to read the book, it turns out to be less boring to him in the end and also more interesting to the better students, without causing excessive trouble for the slow student. Why don't we use textbooks in mathematics which have a high density of ideas? What could it hurt?

Some of the defects of the book as a textbook could be obviously remedied by some worked-out examples, some answers to the problems, and more exercises. The method of paging two parallel accounts could be much improved, though it is more inconvenient to a casual reader than to a student who studies the book.

One of the Committee's general efforts seems to have been fairly well realized. This was an effort to rogue out unessential prerequisites to the study of calculus. An important one of these was the reexamination of the relation of the Euclidean analytic geometry to the calculus. The *Universal Mathematics, Part I* was written with a consistent "graphs" geometry, which is the geometry of the Cartesian product of the real line and the real line without the imposition

of the invariant Euclidean metric in the plane. Perhaps, teachers will feel that this is a doctrinaire position, but it seems in trial to make things actually easier. The modified Menger notation scheme seems to cause no particular difficulty if one does not attempt to explain it, and it may make teachers feel more honest.

There is still a serious difficulty in getting through the foundation of the elementary calculus. The method of writing so far has resulted in an excessive dwelling upon the real numbers system in the beginning although it was the intention of the authors to find a correct but brief foundation for calculus. A simpler start must be found. But despite the over-formality of this writing, the calculus theory presented does represent a real simplification.

At Tulane, as everywhere else, when the students actually get to elementary calculus they find it interesting and comparatively easy. In short, the Tulane experience indicates that elementary calculus is more appropriate for beginning college students than college algebra, or trigonometry, or analytic geometry. The morale of the students is better, and the technical difficulties with it are less than in the conventional subjects. Moreover, there is still reason to hope that students will learn more algebra by studying a well simplified calculus than they will in a course in algebra.

The treatment of the exponential and logarithmic functions before the trigonometric function worked out well. The fact that the calculus was imbedded in the "graphs" geometry dictates a postponement of the study of the trigonometric functions until one returns to Euclidean geometry. This analytic trigonometry can be postponed to the beginning of the second year calculus without causing any difficulty in most programs.

Thus *Universal Mathematics, Part I*, belongs to a growing trend in first year college mathematics to start with calculus as soon as possible and on as simple a foundation as possible. Whether one continues with more calculus and geometry in the second semester or whether one turns to sets and integers, as *Universal Mathematics, Part II*, proposes to do, is a matter to be determined by what is the most valuable in actual experiences, rather than by opinion. The Committee is preparing *Part II* which is formally sets and integers and intuitively "choice and chance" to make a comparison by experience possible.

W. L. DUREN, JR.
Tulane University

Differential and Integral Calculus. By H. M. Bacon. Second Edition. McGraw-Hill Book Company, New York, 1955. vii+547 pages. $6.00.

Calculus. By William L. Hart. D. C. Heath & Company, Boston, 1955. xiii+626 pages. $5.50.

The first edition of Bacon's book was published in 1942 but apparently was not reviewed in this MONTHLY at that time. The author states that several sections have been rewritten for the second edition but he does not claim to have added new material or to have changed the order of topics. This book is typical

of a large number of texts designed for use in traditional introductory courses devoted primarily to the development of formal skills. It deserves to be ranked among the best of its class. While individual sections are well written and show the result of careful planning in the light of much classroom experience, the general organization of the topics could be improved. For example, the presentation of some sixty pages of material on the computation of anti-derivatives before giving any motivation for such activity has obvious disadvantages as does a long chapter on limits placed at the very beginning of the text. (The author mentions in the preface that this chapter on limits can be omitted.) Solid analytic geometry is presupposed and the discussion of partial derivatives and multiple integrals is rather brief. The book closes with a chapter on the elements of differential equations.

Several factors combine to render Hart's book an outstanding addition to the list of calculus texts. In the first place it is more extensive than most books. At least three semesters would be needed to present all of the material which it contains. This is due in part to the fact that in addition to rather full but more or less standard chapters on differential equations and hyperbolic functions there are three chapters devoted to advanced topics in partial differentiation and their applications in which are treated matters such as envelopes, implicit differentiation in general systems, elementary differential geometry of space curves and surfaces, Taylor's series for functions of two variables, *etc.*, which are not often found together in one elementary text. The definite integral is introduced as soon as the usual material on the differential calculus has been discussed, and the student is required to compute some of these by elementary summation and limiting processes before any connection with anti-derivatives is mentioned. The sequence of three chapters on methods of integration is interrupted by a chapter in which the major applications of the definite integral are presented. (The fact that the discussion of centroids and moments of inertia is here confined to the case of one dimensional mass distributions, while the discussion of these topics in the case of two and three dimensional distributions is postponed until multiple integrals are available deserves to be noted and approved.) While plane analytic geometry is a prerequisite, such topics as parametric equations and polar coordinates are treated *ab initio*. In fact, unusual emphasis is placed on the parametric form. This yields some useful results, not the least of which is a most convincing intuitive derivation of L'Hospital's rule. There is an excellent chapter on solid analytic geometry, and the emphasis which has been placed on parametric equations makes the transition to space curves a simple matter. A most conspicuous feature is the steady increase in the level of sophistication as the book progresses. At no stage however, is there hesitation in mentioning results which have proofs beyond the level of the book. When this is done the meaning and use of the theorems mentioned are well illustrated by examples.

H. T. MUHLY
State University of Iowa

OBITUARY

WILLIAM DEWEESE CAIRNS

In Memoriam

William DeWeese Cairns was born in Troy, Ohio, on November 2, 1871, and died on July 15, 1955, in Pasadena, California. In 1898 he married Iva M. Crofoot, and there were two children, Mary Catharine Cairns who died in 1938, and Robert W. Cairns who is Director of Research for the Hercules Powder Company of Wilmington, Delaware. After the death of his first wife he married, in 1930, Mrs. Bertha N. Pope of Oberlin, who survives him. He was a member of the Congregational Church.

After preparation in the Troy High School, Cairns entered Ohio Wesleyan University where he received the A.B. degree in 1892. He taught physics in the Troy High School during the years 1894–96, and then went to Harvard University where he received the A.B. degree in 1897 and a year later, as Schattuck Scholar, the degree A.M. After teaching mathematics for a year at the Calumet (Michigan) High School, he became an Instructor in 1899 at Oberlin College, and in 1904 an Assistant Professor. In the fall of 1905, with a two-year leave of absence from Oberlin, he went to Germany and entered Göttingen University from which he received the Ph.D. degree in 1907. His Göttingen dissertation was entitled "*Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen.*" He returned to Oberlin College where he became Professor of Mathematics in 1920 and served as head of the department from 1920 until his retirement in 1939. In 1943 he taught a pre-meteorological course at the University of New Mexico.

To generations of Oberlin College students he was a patient, friendly, and beloved teacher, but members of the Mathematical Association of America will remember Will Cairns as the enthusiastic guiding spirit of that organization through its early years. He served as its Secretary-Treasurer from the beginning in 1916 until he resigned from the office at the end of 1942. On that occasion the late Earle Raymond Hedrick wrote for this Monthly:*

"During this time its membership has increased to considerably more than twice the eleven hundred charter members; meetings are held in twenty-two different sections throughout the country; and an ambitious program of periodical and book publication, and of sponsorship of other periodicals and of prize competitions, has come to fruition. Throughout this period of increasing usefulness the now retiring Secretary-Treasurer, William DeWeese Cairns, has been an outstanding leader in the determination of policy and in the conduct of all the Association's affairs. In the extent and value of his services to the Association, he is rivalled only by Herbert Ellsworth Slaught (1861–1937), also an

* A remarkably good likeness of Professor Cairns appears as a frontispiece to volume 50 (1943) of this Monthly.

enthusiastic and constant worker for the interests of the Association from the time of its founding. To these two men far more than to any others the Association owes its success and its present strong position."

Unwilling to have Cairns sever his official connection with the organization, the Association immediately in 1943 elected him to a two-year term as President. Then at the annual business meeting in Chicago on November 26, 1944, he was made Honorary President of the Association for life.

Though Professor Cairns' most important contribution to American mathematics was undoubtedly the devoted service which he gave to the Association, he still found some time for other mathematical activity. He was the author of about a dozen papers published in *Annals of Mathematics*, *Bulletin of the American Mathematical Society*, *American Mathematical Monthly*, *Science*, and *Mathematics Teacher*. He was a member of the American Mathematical Society, the Ohio Association of Teachers of Mathematics and Science, and the American Association for the Advancement of Science, and he served for many years as the representative of the Mathematical Association on the Council of the latter organization.

The Association made William DeWeese Cairns an Honorary President "for life," but his death can not terminate our appreciation of his most valuable service, and those of us who had the privilege of knowing him personally may continue to honor his memory.

W. B. CARVER

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SHELL MERIT FELLOWSHIPS FOR HIGH SCHOOL SCIENCE AND MATHEMATICS TEACHERS

The Shell Companies Foundation, Inc., has announced a broad program of recognition fellowships for high school teachers of science and mathematics.

Through the program, worked out with the cooperation of leading educational associations, Shell will underwrite summer seminars at Stanford and Cornell Universities for sixty teachers yearly. The Fellowship recipients, chosen on the basis of merit and demonstrated leadership qualities, will receive travel allowances, all tuition and fees, living expenses on the university campus and $500 in cash to make up for the loss of potential summer earnings. The fellow-

ships were particularly designed for able experienced teachers who ordinarily might seek remunerative summer employment outside the school system. The intensive seminar programs will include graduate-level classes, lectures by outstanding scientists, and visits to modern industrial installations and research laboratories.

Mathematics, physics, or chemistry teachers with five years' experience and known leadership ability will be eligible for the Fellowships. Thirty teachers from west of the Mississippi River will attend the eight-week Stanford program which will be administered by the Stanford School of Education. Thirty from east of the Mississippi will be invited to a similar six-weeks' series of courses at Cornell.

In addition to teachers, present heads of departments or supervisors with good backgrounds in mathematics, chemistry or physics who were previously teachers are also eligible. Full responsibility for the final selection of the sixty will rest with Stanford University and Cornell University.

Requests for Fellowship applications should be sent to Cornell University and Stanford University. The completed form of each applicant must be accompanied by personal recommendations from his high school principal, a faculty member from his most recent college, and someone who can attest to the applicant's leadership talents with young people outside school or with his professional associates.

The Fellowships are in addition to the Shell Companies Foundation's present $350,000 aid-to-education program which includes fifty graduate fellowships and twenty grants in fundamental research in science and engineering at forty-one colleges and universities.

SUMMER INSTITUTES FOR HIGH SCHOOL AND COLLEGE TEACHERS

The National Science Foundation will sponsor three institutes for teachers of mathematics in the summer of 1956. Preliminary announcements of these appear below. In each case housing will be provided in university dormitories and a limited number of stipends will be available for participants.

Iowa State Teachers College. This institute for high school teachers will be held during the period June 18 to July 27 under the direction of Professor Henry Van Engen. Inquiries should be addressed to Professor Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

University of Michigan. This institute for college teachers will be held during the period June 25 to August 14 under the direction of Professor T. H. Hildebrandt with the assistance of Professor P. S. Jones. The principal lecturers will be: Professor C. C. MacDuffee of the University of Wisconsin, whose title is "Elementary Algebra from an Advanced Viewpoint" and Professor G. B. Price of the University of Kansas whose title is "Limits, Derivatives and Integrals." Inquiries should be addressed to Professor Hildebrandt, University of Michigan, Ann Arbor, Michigan.

Williams College. This institute for high school and college teachers will be held during the period July 2 to August 11 under the direction of Professor Donald E. Richmond. The principal speakers will be Professor B. W. Jones of the University of Colorado who will lecture on "Algebra and Number Theory," and Professor S. C. Kleene of the University of Wisconsin whose title is "Sets, Logic, and Mathematical Foundations." Shorter lecture series will be given by R. H. Bing, "Topology"; W. L. Duren, Jr., "Universal Mathematics"; W. Prenowitz, "Geometric Vector Analysis and the Concept of Vector Space," and others, some from industry. Inquiries should be addressed to Professor Richmond, Williams College, Williamstown, Massachusetts.

SUMMER INSTITUTE FOR HIGH SCHOOL TEACHERS OF MATHEMATICS

The University of Minnesota will conduct a program in the basic and applied aspects of mathematics as these are related to the pedagogical needs of high school teachers. The program will be offered by the Department of Mathematics during the first term of the Summer Session. Scholarships covering tuition and living expenses will be available. Six credits in the Graduate School will be given to those who have been admitted to the Graduate School and who satisfactorily complete the work in the program. These credits will be applicable toward advanced degrees for those whose major subject is Education and whose minor subject is Mathematics and for those whose major subject is Mathematics. Those interested in the institute should write to B. R. Gelbaum, Associate Professor of Mathematics, 119 Folwell Hall, University of Minnesota, Minneapolis 14, Minnesota.

PERSONAL ITEMS

Catholic University announces the following: Mr. Eugene Lukacs, previously head of the Statistical Branch, Office of Naval Research, Washington, D. C., has been appointed to a professorship; Assistant Professor Choy-tak Taam has been promoted to an associate professorship; Interim Associate Professor N. A. Wiegmann has been promoted to an associate professorship.

Fenn College reports: Assistant Professor A. W. Brunson of Michigan College of Mining and Technology has been appointed to an assistant professorship; Mr. David Helacher, formerly a graduate assistant at the State University of Iowa, has been appointed to an instructorship.

Pennsylvania State University announces: Dr. Robert Ellis of the University of Chicago, Dr. P. C. Gilmore, formerly a research associate at the University of Toronto, Assistant Professor Harry Gonshor of the University of Miami, and Dr. G. N. Raney of Brooklyn College have been appointed to assistant professorships; Mr. R. L. Duncan, formerly a graduate assistant at the University, has been appointed to an instructorship; Mrs. Catherine Euwema and Miss Audrey Weber, previously a graduate assistant at the University, have been appointed to part-time instructorships.

At Purdue University: Professor Ralph Hull, head of the Department of Mathematics, is on leave of absence as a consultant at the Ramo-Wooldridge Corporation, Los Angeles, California; Professor Arthur Rosenthal is Acting Head of the Department during the academic year 1955–1956; Assistant Professor O. P. Aggarwal of the University of Washington, Dr. Helen Bozivich, formerly a research associate at Iowa State College, and Dr. W. H. Fleming, previously a mathematician at the Rand Corporation, Santa Monica, California, have been appointed to assistant professorships; Assistant Professors P. E. Irick and J. H. B. Kemperman have been promoted to associate professorships; Dr. Robert Baer, Dr. N. D. Kazarinoff, and Dr. Daniel Waterman have been promoted to assistant professorships; Dr. Chester Feldman of the Hammond Extension Center of the University has been promoted to an assistant professorship; Mr. J. E. Forbes, previously a teaching assistant, has been promoted to an instructorship; Professor C. T. Hazard and Associate Professor C. N. Wunder have retired with the title of Professor Emeritus; Assistant Professor Henry Teicher is on leave of absence for the year 1955–1956 and is in the Department of Mathematical Statistics at Stanford University; Dr. Robert Baer and Dr. Daniel Waterman received research grants for the summer of 1955.

University of New Hampshire reports the following: Mr. F. J. Lorenzen, Jr., previously an instructor at Tilton Academy, New Hampshire, has been appointed to a graduate assistantship; Mr. C. W. Schenck, formerly a lecturer at the University, has been appointed to an instructorship in the Mechanical Engineering Department.

Dr. M. Q. Barton, formerly a teaching assistant at the University of Illinois, has been appointed a research associate at Brookhaven National Laboratory, Upton, Long Island, New York.

Dr. J. S. Bendat, previously a research engineer at Northrop Aircraft Company, Hawthorne, California, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor W. H. Carnahan of Purdue University now has a position with D. C. Heath and Company, Publishers.

Professor Emeritus E. W. Chittenden of the State University of Iowa has accepted a position as mathematician with the Diamond Ordnance Fuse Laboratories, Washington, D. C., for the year 1955–1956.

Dr. A. E. Danese of the University of Tennessee has accepted a position as a mathematician with Eastman Kodak Company, Rochester, New York.

Mr. Reid Haywood, formerly a mathematician at the United States Naval Proving Ground, Dahlgren, Virginia, has a position as a senior computation mathematician at the Glenn L. Martin Company, Baltimore, Maryland.

Assistant Professor Jacob Korevaar of the University of Wisconsin has been promoted to an associate professorship.

Mr. G. N. Landes, previously a mathematical analyst at the Sandia Corpora-

tion, Albuquerque, New Mexico, has been appointed an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. Mark Lotkin of the Radio Corporation of America Service Company, Inc., Patrick Air Force Base, Cocoa, Florida, has accepted a position as a mathematician with the Republic Aviation Corporation, Farmingdale, New York.

Mr. W. C. Lowry, previously an assistant instructor at Ohio State University, has been appointed to an assistant professorship in education at the University of Virginia.

Miss Mardell S. Miskowski, formerly a mathematician at Aberdeen Proving Ground, Maryland, has accepted a position as a research engineer at North American Aviation, Downey, California.

Mr. C. R. Perisho of Mankato State Teachers College has been promoted to an assistant professorship.

Dr. C. M. Petty of Purdue University has a position with the Lockheed Missile Division, Van Nuys, California.

Mr. S. L. Prosser, previously a graduate assistant at the University of Alabama, has been appointed an applied science representative with the I.B.M. Corporation, Atlanta, Georgia.

Dr. Anatol Rapoport of the Center for Advanced Study in the Behavioral Sciences, Stanford, California, has been appointed to an associate professorship in mathematical biology at the Mental Health Research Institute, University of Michigan.

Mr. W. G. Stokes of Vanderbilt University has been appointed Associate Professor and Head of the Department of Mathematics at Austin Peay State College.

Dr. R. G. Stoneham, recently a mathematician at Logistics Research, Inc., Redondo Beach, California, has been appointed to an assistant professorship at the University of California, Santa Barbara College.

Mr. Irwin Stoner, formerly a dynamics engineer at the Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as an associate mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Miss Virginia R. Swann, previously a graduate student at the University of North Carolina, is now employed as a mathematician at the Naval Ordnance Laboratory, Silver Spring, White Oak, Maryland.

Mr. L. O. Thompson has been appointed to an instructorship at Marshall College.

Assistant Professor S. I. Vrooman of Rensselaer Polytechnic Institute has been appointed to a visiting professorship at the University of Pittsburgh.

Dr. D. T. Walker has been appointed to an assistant professorship at Memphis State College.

Mr. L. F. Babcock of the Woodmere Academy, Long Island, New York, died on August 27, 1955.

Mr. A. D. Pierson, retired chairman of the Department of Mathematics of the Junior College of Kansas City, died on October 26, 1955. He was a member of the Association for thirty-five years.

Professor Emeritus Mary E. Sinclair of Oberlin College died on June 3, 1955. She was a charter member of the Association.

Professor Emeritus M. O. Tripp of Wittenberg College died on November 30, 1955. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-NINTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-ninth annual meeting of the Mathematical Association of America was held at the Rice Institute, Houston, Texas, on Friday, December 30, 1955, in conjunction with the annual meeting of the American Mathematical Society. Three hundred and eighty-two persons were registered, including two hundred and fifty-four members of the Association.

Sessions of the Association were held on Friday morning and afternoon in the Physics Amphitheatre of Rice Institute. Vice-President G. B. Price presided at the morning session, President W. L. Duren at the annual business meeting, and Professor W. T. Reid presided at the remainder of the afternoon session. The Program Committee for the meeting consisted of Wallace Givens, Chairman; W. T. Reid, and C. R. Sherer.

FIRST SESSION OF THE ASSOCIATION

"Some aspects of the theory of knots," by Dr. Guillermo Torres, Institute of Mathematics, University of Mexico.

"Simple unsolvable problems about semi-groups and groups," by Professor W. W. Boone, Catholic University of America and the Institute for Advanced Study.

"Mathematics instruction and the new fields of electronic computation," by Dr. H. R. J. Grosch, Aircraft Gas Turbine Division, General Electric Company.

SECOND SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"Report on current programs to improve science teaching," by Dr. L. W. Cohen, National Science Foundation.

"Some recent discoveries and near misses in topology," by Professor R. H. Bing, University of Wisconsin.

"Stimulating interest of youthful talent in mathematics and other sciences," by Professor H. J. Ettlinger, University of Texas.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday evening in the lounge of the Cohen House, with eighteen members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Duren of the following Nominating Committee for 1956: Ivan Niven, Chairman; H. E. Bray, and M. R. Hestenes.

Professor W. B. Carver was re-elected as a member of the Finance Committee for 1956–1959.

It was voted to hold the Thirty-eighth Summer Meeting at Pennsylvania State University in August 1957 and to hold the 1957–1958 annual meeting (normally scheduled for December 1957) during the latter part of January 1958.

The Board voted to approve the printing of two ninety-six page issues of the MONTHLY during 1956. The publication of two Slaught papers during 1956 was approved if suitable manuscripts are available. The Editor was instructed to omit from the MONTHLY hereafter the names of members attending meetings of the Association and its sections, giving only the number of persons and of members in attendance.

Professor J. C. Oxtoby of Bryn Mawr College was invited to deliver the Hedrick Lectures on "Category and Measure" at the 1956 Summer Meeting.

On the recommendation of the Committee on Sections, the Board voted to hold meetings of section officers annually, preferably at the time and place of the summer meeting; to authorize the establishment of a Northeastern Section of the Association to include the territory previously included in the New England Region; to transfer Huntington (Cabell County), West Virginia, from the Allegheny Mountain Section to the Ohio Section.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, December 30, 1955, in the Physics Amphitheatre of the Rice Institute, Houston, Texas. President W. L. Duren presided.

The Secretary announced the results of the balloting for officers in which 1635 votes were cast: Professor R. V. Churchill of the University of Michigan was elected First Vice-President for the two-year term 1956–1957, and Dr. A. S. Householder of Oak Ridge National Laboratory and Professor M. F. Smiley of the State University of Iowa were elected Governors for the three-year term 1956–1958.

The Association was organized on this same date forty years ago with a charter membership of 1045 persons. Of these, 208 are still living and have maintained continuous membership in the Association. The membership as of December 23, 1955, is 6045.

A report for the Committee on Visiting Lecturers was presented by Professor D. E. Richmond. Professor J. S. Frame reported for the Committee on Employ-

ment Opportunities that the Employment Register had been available throughout the meeting in Anderson Hall. Dean W. L. Duren described the plans of the Committee on the Undergraduate Program, and Professor G. B. Price asked that suggestions be sent to him as chairman of the Committee to Review the Activities of the Association.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday, December 27 through Thursday morning. The Gibbs lecturer was Professor Joseph E. Mayer. Professor G. T. Whyburn delivered his retiring presidential address and an invited address was delivered by Professor S. S. Chern.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: F. E. Ulrich, Chairman; H. E. Bray, A. H. Brown, L. K. Durst, H. M. Gehman, Guy Johnson, Jr., G. R. MacLane, Szolem Mandelbrojt, and J. W. T. Youngs.

Registration headquarters was in the Fondren Library of Rice Institute. Dormitory accommodations were provided in Wiess Hall and meals were available in the Rice Commons. The lounges of the Fondren Library and of the Cohen House (Faculty Club) were open daily. Tea was served every afternoon in the Cohen House. A banquet was held on Wednesday evening in the Commons. On Thursday afternoon a trip was taken to the Baytown Refinery of the Humble Oil and Refining Company, and on Thursday evening there was a concert by the Lyric Art String Quartet in Fondren Library.

President R. L. Wilder of the Society acted as toastmaster at the banquet. President W. V. Houston of Rice Institute spoke on the critical shortage of mathematicians and other scientists, and President W. L. Duren responded for the Association by telling of some of the steps being taken by the Association to attempt to alleviate this shortage. Dr. L. W. Cohen presented a resolution of thanks which was unanimously adopted expressing the deep appreciation of the members of both mathematical organizations to the Faculty and their ladies, to the President and Trustees of the Rice Institute for the excellent arrangements and accommodations provided for the meeting. Both organizations will remember the characteristic Texas hospitality they enjoyed during the Annual Meeting in 1955.

H. M. GEHMAN, *Secretary-Treasurer*

ITINERARIES OF VISITING LECTURERS, 1956

G. B. Price

Univ. of Arkansas and environs	Fayetteville, Ark.	Feb. 13–22
Memphis State College	Memphis, Tenn.	Feb. 23–25
Denison Univ.	Granville, Ohio	March 12–14
Case Institute of Technology	Cleveland, Ohio	March 15–17

Univ. of Detroit and Wayne Univ.	Detroit, Mich.	March 19–23
Michigan Section, M.A.A.	Ann Arbor, Mich.	March 24
Iowa State Teachers College	Cedar Falls, Iowa	April 2–4
St. Olaf Coll. and Carleton Coll.	Northfield, Minn.	April 5–8
Macalester College	St. Paul, Minn.	April 9–10
Beloit Coll. and Univ. of Wisconsin	Beloit and Madison, Wis.	April 11–12
Nat'l. Council of Teachers of Math.	Milwaukee, Wis.	April 13
Univ. of Omaha	Omaha, Neb.	April 16–18
Univ. of Nebraska	Lincoln, Neb.	April 19–21
State Teachers College	Shippensburg, Pa.	May 7–9
Drew Univ.	Madison, N. J.	May 10–12
Hunter Coll. and environs	New York, N. Y.	May 14–16
Skidmore College	Saratoga Springs, N. Y.	May 17
Alfred Univ.	Alfred, N. Y.	May 18

D. H. Lehmer

San Jose State College	San Jose, Calif.	Feb. 20 or 21
Univ. of Redlands	Redlands, Calif.	Feb. 29–March 2
Whittier College	Whittier, Calif.	March 2
Univ. of Arizona	Tucson, Ariz.	March 5–9
New Mexico A and M	Las Cruces, N. M.	March 12–13
New Mexico Inst. of Mining and Tech.	Socorro, N. M.	March 14–16
Univ. of Colorado and environs	Boulder, Colo.	March 19–22
Texas Christian Univ. and environs	Fort Worth, Tex.	March 26–28
Fresno State College	Fresno, Calif.	April 3
Univ. of Oregon	Eugene, Ore.	April 23–25
Oregon State College	Corvallis, Ore.	April 26–27
Univ. of Washington and environs	Seattle, Wash.	April 30–May 4
Montana State Univ.	Missoula, Mont.	May 7–8
State Coll. of Washington and Univ. of Idaho	Pullman, Wash. and Moscow, Idaho	May 9–12
Northwest Nazarene College	Nampa, Idaho	May 14–15

George Pólya

Tulane Univ.	New Orleans, La.	Feb. 17–21
Philander Smith College	Little Rock, Ark.	Feb. 23–25
Univ. of Mississippi	University, Miss.	Feb. 27–28
Alabama Polytechnic Inst.	Auburn, Ala.	March 1–3
Univ. of Miami	Coral Gables, Fla.	March 5–7
Univ. of Florida	Gainesville, Fla.	March 8–10
Univ. of South Carolina	Columbia, S. C.	March 12–15
Southeastern Section of M.A.A.	Athens, Ga.	March 16–17

The visiting lecturers for 1956–57 will be Professors Edwin Hewitt of the University of Washington, Kurt Mahler of the University of Manchester, England, A. W. Tucker of Princeton University, and J. L. Walsh of Harvard University. It is expected that detailed announcements will be sent out April 15 to all mathematics departments having members of the Association. Anyone not receiving an announcement by early in May may write to the Secretary of the Association for details and application blanks.

OFFICERS AND COMMITTEES AS OF JANUARY 1, 1956

OFFICERS

President, W. L. DUREN, JR., University of Virginia (1955–1956)
First Vice-President, R. V. CHURCHILL, University of Michigan (1956–1957)
Second Vice-President, G. B. PRICE, University of Kansas (1955–1956)
Editor, C. B. ALLENDOERFER, University of Washington (1952–1956)
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo (1953–1957)
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo (1953–1957)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

R. E. LANGER, University of Wisconsin (1951–1956)
SAUNDERS MACLANE, University of Chicago (1953–1958)
E. J. MCSHANE, University of Virginia (1955–1960)

Governors at Large

PHILIP FRANKLIN, Massachusetts Institute of Technology (1954–1956)
C. E. SPRINGER, University of Oklahoma (1954–1956)
H. W. BRINKMANN, Swarthmore College (1955–1957)
M. A. ZORN, Indiana University (1955–1957)
A. S. HOUSEHOLDER, Oak Ridge National Laboratory (1956–1958)
M. F. SMILEY, State University of Iowa (1956–1958)

Sectional Governors (July 1, 1953–June 30, 1956)

Illinois, E. B. MILLER, Illinois College
Iowa, E. N. OBERG, State University of Iowa
Louisiana-Mississippi, F. A. RICKEY, Louisiana State University
Maryland-Dist. of Col.-Virginia, S. B. JACKSON, University of Maryland
Michigan, P. S. JONES, University of Michigan
Minnesota, K. O. MAY, Carleton College
Philadelphia, C. O. OAKLEY, Haverford College
Southern California, C. W. TRIGG, Los Angeles City College
Texas, E. R. HEINEMAN, Texas Technological College

Sectional Governors (July 1, 1954–June 30, 1957)

Allegheny Mountain, MORRIS OSTROFSKY, Westinghouse Electric Corporation
Indiana, RALPH HULL, Purdue University
Kentucky, H. H. DOWNING, University of Kentucky
Metropolitan New York, R. M. FOSTER, Polytechnic Institute of Brooklyn
Nebraska, M. A. BASOCO, University of Nebraska
Northern California, W. H. MYERS, San Jose State College
Oklahoma, L. W. JOHNSON, Oklahoma Agricultural and Mechanical College
Rocky Mountain, C. A. HUTCHINSON, University of Colorado
Wisconsin, R. C. HUFFER, Beloit College

Sectional Governors (July 1, 1955–June 30, 1958)

Kansas, C. B. READ, University of Wichita
Missouri, F. F. HELTON, Central College
Northeastern, G. B. THOMAS, JR., Massachusetts Institute of Technology
Ohio, L. L. LOWENSTEIN, Kent State University
Pacific Northwest, IVAN NIVEN, University of Oregon

Southeastern, F. W. KOKOMOOR, University of Florida
Southwestern, M. S. HENDRICKSON, University of New Mexico
Upper New York State, J. F. RANDOLPH, University of Rochester

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1956–1959), J. F. RANDOLPH (1954–1957), H. M. GEHMAN, *ex officio*.

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

TIBOR RADO, *Chairman* (1954–1959), I. S. SOKOLNIKOFF (1951–1956), P. R. HALMOS (1952–1957), SAMUEL EILENBERG (1953–1958), C. G. LATIMER (1955–1960), E. E. FLOYD (1956–1961).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

EDWIN HEWITT, *Chairman* (1954–1956), MARK KAC (1955–1957), S. C. KLEENE (1956–1958).

JOINT COMMITTEE ON PLACES OF MEETINGS

W. M. WHYBURN, *Chairman* (1954–1956), C. B. MORREY, JR. (1955–1957), R. M. THRALL (1956–1958).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

L. M. KELLY, *Chairman* (July 1954–June 1957), R. E. GREENWOOD (July 1955–June 1956), LEO MOSER (July 1955–June 1958); L. E. BUSH, *Director* (July 1953–June 1958).

COMMITTEE ON SECTIONS

J. C. POLLEY (1953–1956), C. H. FRICK (1955–1958), EDITH R. SCHNECKENBURGER, *ex officio*.

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

J. L. KELLEY, *Chairman* (1954–1956), H. S. M. COXETER (1955–1957), R. D. SCHAFER (1956–1958), C. B. ALLENDOERFER, *ex officio*.

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

J. S. FRAME, *Chairman*, H. M. BACON, RAY BERKOWITZ, W. M. HIRSCH, MORRIS OSTROFSKY, G. W. PATTERSON, J. A. WARD.

COMMITTEE ON FILMS FOR CLASSROOM INSTRUCTION

P. S. JONES, *Chairman*, H. H. CAMPAIGNE, MARGUERITE LEHR.

COMMITTEE ON HIGH SCHOOL CONTESTS

T. F. COPE, *Chairman*, R. P. BAILEY, MAX KRAMER, EDITH R. SCHNECKENBURGER, ROTHWELL STEPHENS.

COMMITTEE ON MATHEMATICAL PERSONNEL AND EDUCATION

TOMLINSON FORT, *Chairman*, E. A. CAMERON, H. P. EVANS, M. GWENETH HUMPHREYS, IVAN NIVEN, E. P. NORTHROP, E. T. WELMERS.

COMMITTEE TO STUDY THE ACTIVITIES OF THE ASSOCIATION

G. B. PRICE, *Chairman*, C. W. CURTIS, DAVID GALE, ROTHWELL STEPHENS, R. R. STOLL, PATRICK SUPPES, G. B. THOMAS, JR.

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

E. J. MCSHANE, *Chairman*, J. G. KEMENY, G. B. PRICE, A. L. PUTNAM, A. W. TUCKER, W. L. DUREN, JR., *ex officio*.

COMMITTEE ON VISITING LECTURERS

B. W. JONES, *Chairman*, G. B. HUFF, D. E. RICHMOND, R. A. ROSENBAUM.

COMMITTEE ON THE 1956 CHAUVENET PRIZE

P. A. WHITE, *Chairman*, J. D. MANCILL, J. G. WENDEL.

NOMINATING COMMITTEE FOR 1956

IVAN NIVEN, *Chairman*, H. E. BRAY, M. R. HESTENES.

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
D. E. RICHMOND (1955–1957), W. L. DUREN, JR., *ex officio*, H. M. GEHMAN, *ex officio*.
On the National Research Council:
EINAR HILLE (July 1, 1953–June 30, 1956).
On the Council of the American Association for the Advancement of Science:
L. M. GRAVES (1955–1956), H. J. ETTLINGER (1956–1957).
On the American Council on Education:
W. L. DUREN, JR., *ex officio*, H. M. GEHMAN, *ex officio*.
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
DAVID BLACKWELL (1954–1956).
On the Committee on Definitions of Electrical Terms:
S. A. SCHELKUNOFF.
On a Committee of the N.C.T.M. on a Vocational Pamphlet:
A. L. PUTNAM.
Associate Editor of Mathematics Student Journal:
IZAAK WIRSZUP (1954–1956).

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.
No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.
No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.
No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1956.

MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

OKLAHOMA, University of Tulsa, March 31, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April 20–21, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. The following four numbers have already been published:

1. *Fourier Series; The Genesis and Evolution of a Theory* by R. E. Langer. v+86 pages.
2. *Outline of the History of Mathematics* (6th edition) by R. C. Archibald. iv+114 pages.
3. *Proceedings of the Symposium on Special Topics in Applied Mathematics.* iv+73 pages.
4. *Contributions to Geometry.* iv+75 pages.

Copies at one dollar each postpaid may be ordered from: Professor Harry M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

RCA offers opportunities

IN MISSILE TEST

Data Reduction

for

- MATHEMATICIANS
- STATISTICIANS
- PHYSICISTS
- ASTRO-PHYSICISTS

Degree plus experience in reduction of test data, applied mathematics, statistical techniques, or observatory practices. Positions now available on Florida's central east coast.

Liberal company benefits—Relocation assistance.

For information and arrangements for personal interview, send complete resume to:

Mr. D. E. Pinholster
Employment Manager, Dept. N-11C
Missile Test Project
RCA Service Co., Inc.
P.O. Box 1226
Melbourne, Florida

RADIO CORPORATION OF AMERICA

NEW REPRINT

Ready Spring 1956

American Mathematical Monthly

Volumes 1-20, 1894-1913

Cloth bound set	$275.00	
Paper bound set	245.00	
Volumes 1-9, paper bound		$15.00 each
Volumes 10-20, paper bound		10.00 each

This reprint is being undertaken with the permission of the Mathematical Association of America. Please address orders and inquiries to

JOHNSON REPRINT CORPORATION
125 East 23 Street
New York 10, New York

FOR THE MATHEMATICIAN
who's ahead of his time

IBM is looking for a special kind of mathematician, and will pay especially well for his abilities.

This man is a pioneer, an educator—with a major or graduate degree in Mathematics, Physics, or Engineering with Applied Mathematics equivalent.

You may be the man.

If you can qualify, you'll work as a special representative of IBM's Applied Science Division, as a top-level consultant to business executives, government officials and scientists. It is an exciting position, crammed with interest, and responsibility.

Employment assignment can probably be made in almost any major U. S. city you choose. Excellent working conditions and employee-benefit program.

For applicants with the same basic qualifications, opportunities are available to teach in this exciting, new field.

Your reply will, of course, be held in the strictest confidence. Write, giving full details of education and experience, to:

Dr. C. R. DeCarlo
DIRECTOR, APPLIED SCIENCE DIVISION

INTERNATIONAL BUSINESS MACHINES CORP.
590 Madison Avenue, New York 22, N. Y.

Hundreds of IBM electronic data processing machines are already in use, and many more will be installed during 1956. These data processing machines have become essential for computations in science, engineering, and business management.

IBM

Producer of electronic data processing machines, electric typewriters, and electronic time equipment.

New

COLLEGE ALGEBRA, Revised

E. A. Cameron and E. T. Browne

Highly successful in the earlier edition, this revision presents the topics of college algebra with as much rigor and precision as is consistent with maintaining student interest. Features new material on statistics, ample exercises and problems, increased emphasis on applications to physics and chemistry.

March 1956

INTRODUCTION TO MATHEMATICS

Lee Emerson Boyer

A stimulating, readable account of the development of mathematics from earliest times to the present. Covers the most important and useful topics of arithmetic, algebra, geometry and trigonometry. New material on installment buying, stocks and bonds, introductory algebra.

January 1956

and recent

Brixey · Andree	**MODERN TRIGONOMETRY**
Brixey · Andree	**FUNDAMENTALS OF COLLEGE MATHEMATICS**
Sisam · Atchison	**ANALYTIC GEOMETRY, Third Ed.**
Hill · Linker	**INTRODUCTION TO COLLEGE MATHEMATICS, Rev.**
E. G. Begle	**INTRODUCTORY CALCULUS With Analytic Geometry**
G. M. Merriman	**CALCULUS**

Henry Holt and Co.

New York 17
San Francisco 5

The leading MATHEMATICS TEXTS

for courses offered to science and engineering students

CURRENT ADOPTIONS

CALCULUS or CALCULUS AND ANALYTIC GEOMETRY

by GEORGE B. THOMAS, JR., *Massachusetts Institute of Technology*

University of Alberta
American University of Beirut (Lebanon)
Antioch College
Bard College
University of British Columbia
Brown University
Bucknell University
University of California (Berkeley)
University of California at L. A.
Carleton College
Case Institute of Technology
Casper College
Catholic University of America
Christian Brothers College
City College of New York
University of Connecticut
Cornell University
East Los Angeles Junior College
Ecole Polytechnique (Montreal)
Fisk University
Florida Southern College
Fullerton Junior College
General Electric Company, N. Y.
Georgetown University
Gonzaga University
Graceland College
Hanover College
University of Havana
Hunter College
University of Illinois
Institute Technologico de Monterrey (Mexico)
Johns Hopkins University
King College
Lafayette College
Laval University
Lawrence College
Loras College
Lowell Technological Institute
Manhattan College
College of Marin
Massachusetts Institute of Technology
University of Michigan
Monterey College
Mt. San Antonio College
New England College
New York University
Northwestern University
University of Omaha
Olympic College
Pacific Union College
Polytechnic Institute of Brooklyn
Princeton University
Rensselaer Polytechnic Institute
Rice Institute
Rockhurst College
San Benito County Junior College
St. Josephs College
St. Peters College
Syracuse University
Trinity College (D.C.)
Tulane University
Vallejo College
Valparaiso University
Vanderbilt University
Vassar College
Virginia Military Institute
University of Virginia
Whittier College
University of Wisconsin
Xavier University (Ohio)
Yale University

CALCULUS — *614 pp, 333 illus, 1st ed., 2nd ptg, 1955* — *$7.50*

CALCULUS AND ANALYTIC GEOMETRY — *731 pp, 418 illus, 2nd ed., 3rd ptg, 1955* — *$8.50*

ADVANCED CALCULUS

by WILFRED KAPLAN, *University of Michigan*

Abilene Christian College
University of Alaska
Arizona State College
University of Arizona
Ball State Teachers College
Brandeis University
Bucknell University
University of California (Berkeley)
Carnegie Institute of Technology
Catholic University
City College of New York
Clemson Agricultural College
Colorado College
Columbia University
Cornell University
Dartmouth College
Davidson College
Deep Springs College
Ecole Polytechnique (Montreal)
Emory & Henry College
Florida State University
Georgetown University
Grinnell College
Harvard University
Holy Cross College
Howard University
University of Illinois
Johns Hopkins University
Lafayette College
LaSalle College
Louisiana State University
University of Michigan
Middlebury College
University of Minnesota
University of New Hampshire
University of New Mexico
New York University
University of Notre Dame
Oakland Junior College
University of Oklahoma
Pacific Union College
University of Pennsylvania
Pennsylvania State University
University of Pittsburgh
Polytechnic Institute of Brooklyn
Princeton University
Regis College
University of Rochester
Rutgers University
South Dakota College of A & M
College of South Jersey
Southwest Louisiana Institute
Swarthmore College
Syracuse University
University of Texas
Texas Lutheran College
Trinity College
Trinity University
Tufts University
Tulane University
Union College
Utica College
University of Vermont
University of Virginia
Washington University
Waynesburg College
Whitman College
Whittier College
Willamette University
Williams College
University of Wisconsin
College of Wooster
Xavier College
Yale University

679 pp, 241 illus, 1st ed., 2nd ptg, 1953 — *$9.00*

ADDISON-WESLEY PUBLISHING COMPANY, Inc. Cambridge 42, Mass.

Prentice-Hall
COLLEGE TEXTS

ADVANCED CALCULUS

by DAVID V. WIDDER, Harvard University

Although a student using this text is expected to have considerable skill in the manipulations of elementary calculus, it is not assumed that he will be very familiar with the theoretic side of the subject. Therefore, the book first emphasizes the type of manipulative problem the student has been accustomed to, gradually shifting to the more theoretic problems.

This same method appears in the chapters themselves. In some cases, for example, a fundamental theorem, the meaning of which is easily understood, is stated and used at the beginning of the chapter, while its proof is deferred to the end.

432 pages 6″ x 9″ Published 1947

HANDBOOK OF INDUSTRIAL STATISTICS

by ALBERT H. BOWKER and GERALD J. LIEBERMAN, Stanford University

This book covers statistical quality control, sampling inspection, the design and analysis of experiments, basic concepts, common significance tests, curve fitting, analysis of variance and analysis of enumeration data.

Special attention is paid to methods of making decisions with preassigned risks on the basis of the limited information afforded by samples. These methods are illustrated by examples.

192 pages 6″ x 9″ August, 1955

METHODS OF APPLIED MATHEMATICS

by FRANCIS B. HILDEBRAND, Massachusetts Institute of Technology

This text applies advanced mathematical principles to the solution of engineering problems. In each of the four chapters, the approach consists of: 1) showing how certain types of problems may arise; 2) establishing those parts of the relevant theory which are of practical significance; 3) developing techniques for obtaining exact or approximate solutions to the problems involved.

523 pages 5½″ x 8¼″ Published 1952

ADVANCED CALCULUS FOR ENGINEERS

by FRANCIS B. HILDEBRAND, Massachusetts Institute of Technology

Here is an integrated presentation of special topics and useful methods of calculus essential to engineers and physicists who desire to understand and appreciate modern developments in their fields. This text is also useful to mathematicians who wish to apply their mathematics personally, or to teach others who may do so.

594 pages 5½″ x 8½″ Published 1949

For approval copies write
Prentice-Hall, Inc.

Soon

you* will receive your examination copy of

Britton's CALCULUS

** If you have not already requested a copy, send for one now.*

You will be interested particularly in

- The organization of this text—a carefully planned presentation of differentiation and integration as nearly side-by-side as logical development permits.
- The generous supply of carefully graded problems—including those of a theoretical nature.
- The review questions—designed to develop the student's critical and analytical faculties at the same time that they test his understanding.

Written by Professor Jack R. Britton, the author of several other highly successful texts in college mathematics, this new CALCULUS is the result of many years' thought and teaching experience. Without sacrificing mathematical rigor, it presents an exceptionally clear, easy-to-grasp and thorough introduction to the calculus.

In press

Miller's ENGINEERING MATHEMATICS

A modern, thorough treatment of differential equations, network theory and random functions by a mathematician widely experienced in current engineering research. Includes much material needed in today's engineering development projects that has not heretofore appeared in similar textbooks.

Rinehart (R) Texts in Mathematics

Under the expert supervision of CARROL V. NEWSOM, Executive Vice-Chancellor, New York University, Rinehart's steadily growing mathematics list presents a distinguished group of texts noted for their mathematically modern point of view, pedagogical excellence, and beauty of design. Send for our latest catalog.

RINEHART & COMPANY, INC. **232 Madison Avenue, New York 16**

E. J. CAMP

Mathematical Analysis

A new text containing an integrated treatment of topics from college algebra, trigonometry, analytic geometry, and calculus for the freshman year. Provides a sound foundation for later courses in mathematics. 624 pages. $6.00.*

Texts by WILLIAM L. HART

Calculus

A modern presentation, rigorous, yet skillfully adapted to student understanding. Organization, textual exposition, examples, exercises, and problems of widely varied application make this a teaching instrument of the highest quality. 639p. (558p. text). $5.50*

College Algebra, Fourth Edition

Contains several unusual features—a novel introduction to signed numbers, a unique approach to probability not based on "equally likely" cases, appendix notes on sets, with probability contacts. 490p. (420p. text). $4.00*

Intermediate Algebra for Colleges

Provides a substitute for third-semester algebra. Specifically designed for students who will take a regular algebra course later, or for students who need algebra as a prerequisite for elementary courses in the social sciences or business administration. 323p. (276p. text). $3.75*

**Answers for odd-numbered problems included in the back of the book. Answer book for even-numbered problems available free.*

Home Office: Boston 16
Sales Offices: Englewood, N.J., Chicago 16, San Francisco 5, Atlanta 3, Dallas 1

D. C. HEATH
AND COMPANY

spring 1956 publications

R. S. Underwood
Fred W. Sparks

ANALYTIC GEOMETRY

SECOND EDITION
a revision of tested innovations

R. S. Underwood
H. E. Underwood

PRACTICAL TRIGONOMETRY

all the topics of the classical
trigonometry course pared down
to their essentials

Houghton Mifflin Company

Just Published

INTERMEDIATE ANALYSIS

AN INTRODUCTION TO THE THEORY OF THE FUNCTIONS OF ONE REAL VARIABLE

By JOHN M. H. OLMSTED. This new book is designed for courses in mathematical analysis or the theory of functions which presuppose only one year of calculus. Both the text and the exercises are so arranged as to make the book adaptable both to beginning and advanced students in modern analysis, and to longer or shorter courses.

The book contains 1336 carefully chosen and well arranged exercises, the answers to all of which are provided. Great care has been taken to avoid inexactness and ambiguous statement, and the text contains numerous illustrative examples.

APPLETON
CENTURY
CROFTS

35 West 32nd St.
New York 1, N.Y.

Three noteworthy mathematics texts

Calculus: A Modern Approach

By Karl J. Menger, Illinois Institute of Technology

This text introduces a new method of systematization and clarification of the foundations of analysis in order to reduce the effort needed to master calculus.

Advanced Calculus

By Angus E. Taylor, University of California, L.A.

A thorough treatment with emphasis on sound understanding of concepts and basic principles of analysis. Nice balance is maintained between theory and problems and applications.

Differential Equations

By Frederick H. Steen, Allegheny College

A one-semester or full year's course devoted principally to methods of solving differential equations and to their applications. Designed for a maximum of self-study.

Write for descriptive circulars.

from

GINN AND COMPANY

HOME OFFICE: Boston
SALES OFFICES: New York 11
Chicago 6 Atlanta 3
Dallas 1 Columbus 16
San Francisco 3 Toronto 7

5 important mathematical texts

BEGINNING, INTERMEDIATE AND ADVANCED

First Year Mathematics for Colleges

by PAUL R. RIDER, Professor Emeritus, Washington University

Written in response to wide-spread demands, this single volume of the basic topics for the first year course in mathematics, features simple language and presentation *plus* a generous number of problems accompanying each topic.

1949 *714 pages* *$5.00*

Trigonometry

by JOHN F. RANDOLPH, University of Rochester

This flexible text for plane trigonometry courses features trigonometric concepts and their applications; it reviews elementary algebra, pertinent principles of analytic geometry, and logarithms in the appendices.

1953 *220 pages* *$3.00*

Analytic Geometry, Fifth Edition

by CLYDE E. LOVE and EARL D. RAINVILLE, both at the University of Michigan

Retaining the important successful features of previous editions, the fifth edition treats such new topics as: the distance formula in polar coordinates, circles of Appolonius, radical axis, common chord, generation of surfaces of revolution.

1955 *302 pages* *$4.00*

Differential and Integral Calculus, Fifth Edition

by CLYDE E. LOVE and EARL D. RAINVILLE, both at the University of Michigan

Incorporating new features and improvements, the fifth edition of this famous text contains 3900 exercises (2400 new) many new important topics, and a new treatment of the section on Newton's method for solution of equations.

1954 *526 pages* *$5.75*

Methods in Numerical Analysis

by KAJ L. NIELSEN, Head of the Mathematics Division of the United States Naval Ordnance Plant, Indianapolis

Incorporating recent developments and modern methods, this text details the two major portions of numerical analysis: (1) the analysis of tabulated data, and (2) the numerical methods of finding the solutions to equations.

1956 *382 pages* *$6.90*

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N.Y.

Timely McGRAW-HILL *Books*

PRINCIPLES OF MATHEMATICS

By C. B. ALLENDOERFER, University of Washington; and C. O. OAKLEY, Haverford College. 466 pages, $5.00

A new approach in both content and emphasis directs this important text toward the reform of the basic curriculum in mathematics. The emphasis is on an understanding of the methods of mathematical reasoning, the basic ideas of the subject, and a clear understanding of the reasons behind the mathematical processes. For students who have completed a course in intermediate algebra and who need preparation for a standard calculus course.

MATHEMATICS OF FINANCE. *New Second Edition*

By PAUL M. HUMMEL and CHARLES L. SEEBECK, Jr., University of Alabama. 390 pages, $4.75

Thoroughly covers the mathematics involved in finance and investments, including simple interest, bank discount, compound interest, annuities, perpetuities, sinking funds, amortization of debts, installment buying, bonds, depreciation, life insurance, and life annuities. This new edition has been made more complete and brought fully up to date.

ESSENTIAL BUSINESS MATHEMATICS

By L. R. SNYDER, City College of San Francisco. Second Edition. 421 pages, $4.50

A college text in business mathematics for those who want to develop accuracy and speed in the business processes of fractions, decimals, measurements, and percentages. No knowledge of algebra is necessary. The WORKBOOK accompanying this new edition has also been revised and brought up to date (160 pages, $2.50).

ELEMENTS OF BUSINESS MATHEMATICS FOR COLLEGES

By L. R. SNYDER, 264 pages, $3.75

Presents a sound review of arithmetic and its practical application in solving the most frequently occurring types of business and of personal finance problems. Especially practical for those with only a fair background in high school mathematics. Includes *only* problems that occur constantly in business and personal life. A student workbook with 67 assignments is available.

PLANE TRIGONOMETRY

By C. R. WYLIE, JR., University of Utah. 392 pages, $4.00

Focused primarily on the analytical aspects of plane trigonometry indicating its importance in a subsequent study of mathematics, physics, or engineering. Includes a discussion of complex numbers, trigonometric series, and hyperbolic functions. A glossary of terms and a basic review of geometry and algebra are in the appendix.

CALCULUS

By C. R. WYLIE, JR. 554 pages, $6.00

Covers all material usually taught in a first-term calculus course, including a review of solid analytic geometry and an introduction to ordinary differential equations. Introductory and summary paragraphs in each chapter make the material easy to follow.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN